Sehgal - type contractions in probabilistic metric spaces

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Definition 1.1

A mapping $T : [0,1] \times [0,1] \longrightarrow [0,1]$ *is a triangular norm if T satisfies the following conditions:*

(i) T is commutative and associative;
(ii) $T(a,1) = a$ *for all* $a \in [0,1]$*;*
(iii) $T(a,b) \leq T(c,d)$ *whenever* $a \leq c$ *and* $b \leq d$*.*

Examples: - the Łukasiewicz t-norm $T_L(a,b) = Max(a+b-1,0)$, and the t-norms $T_P(a,b) := ab$, $T_M(a,b) := Min\{a,b\}$

- g-convergent t-norms ([Hadžić, Pap, Budinčević, 2002]):

$$\forall\ q \in (0,1),\ \lim_{n\to\infty} T_{i=n}^{\infty}(1-q^i) = 1.$$

Let Δ_+ denote the space of all functions $F : \mathbb{R} \to [0, 1]$, such that F is left-continuous and non-decreasing on $\mathbb{R}$, and $F(0) = 0$. Let D_+ denote the subclass of Δ containing the functions F with the property $\lim_{t\to\infty} F(t) = 1$.

Definition 1.2

A Menger space is a triple (X, F, T), where X is a nonempty set, T is a t-norm, and $F : X \times X \to D_+$ is a mapping satisfying

(i) $F_{xy}(t) = 1$ *for all* $t > 0$ *if and only if* $x = y$*;*

(ii) $F_{xy} = F_{yx}$*, for all* $x, y \in X$*;*

(iii) $F_{xy}(t + s) \geq T(F_{xz}(t), F_{zy}(s))$*,* $\forall x, y, z \in X$*,* $\forall t, s > 0$*.*

If the mapping F in Definition 1.2 takes values in Δ_+ instead of D_+, (X, F, T) is called a generalized Menger space.

If (X, F, T) is a generalized Menger space with $\sup_{t<1} T(t,t) = 1$, then the family $\{U_{\varepsilon,\lambda}\}_{\varepsilon>0,\lambda\in(0,1),}$ where

$$U_{\varepsilon,\lambda} := \{(x,y) \in X \times X : F_{xy}(\varepsilon) > 1 - \lambda\},$$

is a base for a metrizable uniformity on X, named F - uniformity.

A sequence $(x_n)_n \in X$ is said to be

(i) Cauchy if, for any $\varepsilon > 0$, $\lambda \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $F_{x_n x_m}(\varepsilon) > 1 - \lambda$, for all $n \geq n_0$, $m \geq 1$.

(ii) convergent to $x \in X$ if, for any $\varepsilon > 0$, $\lambda \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $F_{x_n x}(\varepsilon) > 1 - \lambda$ for all $n \geq n_0$.

Definition 2.1 ([Sehgal, 1966])

A Sehgal contraction (or B-contraction) on a Menger space (X, F, T) is a mapping $f : X \to X$ with the property that there exists $k \in (0, 1)$ such that

$$F_{f(x)f(y)}(kt) \geq F_{xy}(t),\ \forall x, y \in X, t > 0. \qquad (2.1)$$

Definition 2.2 ([Miheţ, 1997])

An $(\varepsilon - \lambda)$ - contraction is a mapping f from a Menger space (X, F, T) to itself having the property that there exists $k \in (0, 1)$ such that, for all $\varepsilon > 0, \lambda \in (0, 1)$:

$$F_{xy}(\varepsilon) > 1 - \lambda \Rightarrow F_{f(x)f(y)}(k\varepsilon) > 1 - k\lambda. \qquad (2.2)$$

Definition 2.3 ([Došenović et al, 2011])

Let (X, F, T) be a Menger space, and $(b_n)_n$ be a sequence in $(0, 1)$ such that $b_n \nearrow 1$. A mapping $f : X \to X$ is said to be a strong (b_n) - contraction if there exists $k \in (0, 1)$ such that, for all $n \in \mathbb{N}$, $\varepsilon > 0$, $x, y \in X$,

$$F_{xy}(t) > b_n \Rightarrow F_{f(x)f(y)}(kt) > b_{n+1}. \qquad (2.3)$$

$$\begin{array}{ccc} (\varepsilon, \lambda) - \text{contraction} & \rightarrow & B - \text{contraction} \\ & \searrow & \not\Updownarrow \\ & & \text{strong } (b_n) - \text{ contraction} \end{array}$$

Example 2.1

Let $X = [0, \infty)$ and $F_{xy}(t) = \frac{\min(x,y)}{\max(x,y)}$, for all $x, y \in X$, $x \neq y$, $t > 0$. Then (X, F, T_P) is a complete generalized Menger space, and the mapping $f : X \to X$, $f(x) = x$ is a Sehgal contraction, but it is not a strong (b_n) - contraction.

Example 2.2

We consider $X = \{x, y, z\}$ and the probabilistic metric F on X defined by

$$F_{xz}(t) = F_{zx}(t) = F_{yz}(t) = F_{zy}(t) = \begin{cases} 0, & t \leq 0 \\ \frac{3}{4}, & t \in (0, 2] \\ 1, & t > 2, \end{cases}$$

$$F_{xy}(t) = F_{yx}(t) = \begin{cases} 0, & t \leq 0 \\ \frac{1}{2}, & t \in (0, \frac{3}{2}] \\ 1, & t > \frac{3}{2}. \end{cases}$$

Let $f : X \to X$ be defined by $f(x) = f(y) = x$, $f(z) = y$. Then f is not a Sehgal contraction, but it is a strong (b_n) - contraction with $b_n = 1 - \frac{1}{4^n}$ $(n \geq 1)$.

Theorem 3.1

Let (X, F, T_L) be a complete generalized Menger space, and $(b_n)_n$ be a sequence in $(0,1)$ with $b_n \nearrow 1$, such that $s = \sum_{n=1}^{\infty}(1 - b_n) < \infty$. Let $f : X \to X$ be a strong (b_n) - contraction, with contraction constant $k \in (0,1)$. If there exists $x \in X$ with $F_{xf(x)} \in D_+$, then f has a fixed point $x^* = \lim_{m\to\infty} f^m(x)$. Additionally, if $t > 0$ and b_n are such that $F_{xf(x)}(t) > b_n$, then

$$F_{xx^*}\left(\frac{t}{1-k} + 0\right) \geq \max\{n - s - \sum_{i=1}^{n-1} b_i, 0\}. \tag{3.1}$$

Proof.
Let $\varepsilon > 0$ and $\lambda \in (0, 1)$. It is easy to verify the condition $\sum_{n=1}^{\infty}(1 - b_n) < \infty$ is equivalent to $\lim_{n\to\infty}(T_L)_{i=1}^{\infty} b_{n+i} = 1$, hence one can choose $n \in \mathbb{N}^*$ such that $(T_L)_{i=0}^{\infty} b_{n+i} > 1 - \lambda$. If $F_{xf(x)} \in D_+$, then there exists $t > 0$ with $F_{xf(x)}(t) > b_n$. Inductively, it follows that, for all $m \geq 0$,

$$F_{f^m(x)f^{m+1}(x)}(k^m t) > b_{n+m}.$$

Let $r \geq 1$. We obtain

$$F_{f^m(x)f^{m+r}(x)}\left(\frac{k^m}{1-k}t\right) \geq (T_L)_{i=m}^{m+r-1} b_{n+i} \geq (T_L)_{i=0}^{\infty} b_{n+i}. \quad (3.2)$$

Let $m_0 \in \mathbb{N}$ be such that $\frac{k^{m_0}}{1-k}t < \varepsilon$. Then, for all $m \geq m_0$, $r \geq 1$,

$$F_{f^m(x)f^{m+s}(x)}(\varepsilon) \geq F_{f^m(x)f^{m+s}(x)}\left(\frac{k^m}{1-k}t\right) \geq (T_L)_{i=0}^{\infty} b_{n+i} > 1 - \lambda.$$

Thus $(f^m(x))_m$ is a Cauchy sequence, so it converges to some $x^* \in X$. Since f is continuous, x^* is a fixed point of f.

By setting $m = 0$ in (3.2),

$$F_{xf^r(x)}\left(\frac{t}{1-k}\right) \geq (T_L)_{i=0}^{\infty} b_{n+i} = \max\{n - s - \sum_{i=1}^{n-1} b_i, 0\},$$

for all $r \geq 1$.

For any $\delta > 0$,

$$\begin{aligned} F_{xx^*}\left(\frac{t}{1-k}+\delta\right) &\geq T_L\left(F_{xf^r(x)}\left(\frac{t}{1-k}\right), F_{f^r(x)x^*}(\delta)\right) \\ &\geq T_L(\max\{n-s-\sum_{i=1}^{n-1} b_i, 0\}, F_{f^r(x)x^*}(\delta)) \\ &\xrightarrow{r\to\infty} \max\{n-s-\sum_{i=1}^{n-1} b_i, 0\}, \end{aligned}$$

so the estimation (3.1) holds.

Corollary 3.2

Let (X, F, T_L) be a complete generalized Menger space, and let $f : X \to X$ be a mapping for which there exists $k \in (0, 1)$ such that, for all $\varepsilon > 0$ and $\lambda \in (0, 1)$,

$$F_{xy}(\varepsilon) > 1 - \lambda \Rightarrow F_{f(x)f(y)}(k\varepsilon) > 1 - k\lambda. \qquad (3.3)$$

Suppose there exist $\varepsilon > 0$, $\lambda \in (0, 1)$ and $x \in X$ with $F_{xf(x)}(\varepsilon) > 1 - \lambda$. Then the mapping f has a fixed point x^*, and

$$F_{xx^*}\left(\frac{\varepsilon}{1-k} + 0\right) \geq \max\left\{1 - \frac{\lambda}{1-k}, 0\right\}. \qquad (3.4)$$

We will study the stability of the functional equation

$$f(x) = \Phi(x, f(\eta(x))). \tag{4.1}$$

where the unknown f is a mapping from a nonempty set S to a generalized Menger space (X, F, T_L), and $\eta : S \to S$ and $\Phi : S \times X \to X$ are given mappings.

Theorem 4.1

Let S be a nonempty set, and (X, F, T_L) be a complete generalized Menger space. Suppose that $\Phi : S \times X \to X$ is a mapping for which there exists $k \in (0, 1)$ so that, for all $\varepsilon > 0$ and $\lambda \in (0, 1)$,

$$F_{xy}(\varepsilon) > 1 - \lambda \Rightarrow F_{\Phi(u,x)\Phi(u,y)}(k\varepsilon) > 1 - k\lambda, \ \forall u \in S. \tag{4.2}$$

Then, for every $f : S \to X$ having the property that, for some $\varepsilon > 0$ and $\lambda \in (0, 1)$,

$$F_{f(u)\Phi(u,f(\eta(u)))}(\varepsilon) > 1 - \lambda, \ \forall u \in S, \tag{4.3}$$

there exists a mapping $a : S \to X$ satisfying the equation (4.1)*, with*

$$F_{f(u)a(u)}\left(\frac{\varepsilon}{1-k} + 0\right) \geq \max\left\{1 - \frac{\lambda}{1-k}, 0\right\}, \ \forall u \in S. \tag{4.4}$$

Proof.

Step 1: We consider the space $Y = \{g : S \to X\}$ and Baker's operator $J : Y \to Y$ given by $J(g)(u) = \Phi(u, g(\eta(u)))$, for all $g \in Y$ and all $u \in S$. We define the mapping $\mathcal{F} : Y \times Y \to D_+$ by

$$\mathcal{F}_{gh}(t) = \sup_{s<t} \inf_{u \in S} F_{g(u)h(u)}(s),$$

for all $g, h \in Y$. From the hypotheses on (X, F, T_L), we infer that $(Y, \mathcal{F}, T_L)$ is a complete generalized Menger space.

Step 2: We show that J is an $(\varepsilon - \lambda)$ - contraction on Y.

Step 3: Let f be a mapping satisfying (4.3), for some given $\varepsilon > 0$ and $\lambda \in (0, 1)$. We show that, for any $\delta > 0$,

$$\mathcal{F}_{fJ(f)}(\varepsilon) > 1 - (\lambda + \delta).$$

One can now apply Corollary 3.2 to obtain that the operator J has a fixed point a in Y, meaning that the mapping $a : S \to X$ is an exact solution of (4.1). Moreover,

$$\mathcal{F}_{fa}\left(\frac{\varepsilon}{1-k} + 0\right) \geq \max\left\{1 - \frac{\lambda + \delta}{1-k}, 0\right\}.$$

Since $\delta > 0$ is arbitrary, we obtain the estimation (4.4).

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Thank you for your attention!

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