NEW SZÉCHENYI PLAN

Maps on density operators preserving quantum f-divergences

Patrícia Szokol

joint work with Lajos Molnár and Gergő Nagy

The International Workshop on Functional Analysis

Timisoara (Romania), October 12–14, 2012

*The work is supported by the TÁMOP-4.2.2/B-10/1-2010-0024 project. The project is co-financed by the European Union and the European Social Fund.

Introduction

Classical f-divergences:

- introduced by Csiszár for convex functions
- measure the distance between probability distributions
- widely used concept in information theory and statistics

Introduction

Classical f-divergences:

- introduced by Csiszár for convex functions
- measure the distance between probability distributions
- widely used concept in information theory and statistics

Quantum f-divergences:

- introduced by Petz, Hiai, Mosonyi and Bény
- correspond to classical f-divergences
- measure the distance between quantum states (or density operators)

Introduction

H: finite dimensional complex Hilbert space

- $B(H)$: linear operators on H;
- $B(H)^+$: positive operators on H;
- $S(H)$: positive operators on H with trace 1, density operators;
- $P_1(H)$: rank-one projections on H;

f-divergence

Classical f-divergence

Let $f : [0, \infty) \to \mathbb{R}$ be a convex function and let $P = (p_1, \cdots, p_n)$; $Q = (q_1, \cdots, q_n)$ be probability distributions. Then,

$$D_f(P\|Q) = \sum_{\{i: q_i \neq 0\}} q_i f\left(\frac{p_i}{q_i}\right) + \alpha \sum_{\{i: q_i = 0\}} p_i,$$

where $\alpha = \lim_{x \to \infty} \frac{f(x)}{x}$.

Quantum f-divergence

If $A, B \in B(H)^+$ are given by $A = \sum_{a \in \sigma(A)} aP_a$; $B = \sum_{b \in \sigma(B)} bQ_b$, then

$$S_f(A||B) = \sum_{a \in \sigma(A)} \left(\sum_{b \in \sigma(B) \setminus \{0\}} bf\left(\frac{a}{b}\right) \operatorname{tr} P_a Q_b + \alpha a \operatorname{tr} P_a Q_0 \right).$$

The convenient $0 \cdot (-\infty) = 0 \cdot \infty = 0$ is used.

Examples

1. If $f(x) = x \log_2 x \ (x > 0)$ and $f(0) = 0$, then

$$S_f(A\|B) = S(A\|B) = \begin{cases} \operatorname{tr} A(\log_2 A - \log_2 B), & \operatorname{supp} A \subset \operatorname{supp} B \\ \infty, & \text{otherwise,} \end{cases}$$

which is the Umegaki relative entropy between A and B.

Examples

1. If $f(x) = x \log_2 x$ $(x > 0)$ and $f(0) = 0$, then

$$S_f(A\|B) = S(A\|B) = \begin{cases} \operatorname{tr} A(\log_2 A - \log_2 B), & \operatorname{supp} A \subset \operatorname{supp} B \\ \infty, & \text{otherwise,} \end{cases}$$

   which is the Umegaki relative entropy between A and B.

2. Let $q \in]0,1[$ and the functions $f_q(x) = \frac{x^q}{q-1} - 1$ $(x > 0)$ and $f_q(0) = 0$. Then,

$$S_{f_q}(A\|B) = \frac{1 - \operatorname{tr} A^q B^{1-q}}{1-q},$$

   which is the quantum Tsallis relative entropy between A and B.

Examples

1. If $f(x) = x\log_2 x$ $(x > 0)$ and $f(0) = 0$, then

$$S_f(A\|B) = S(A\|B) = \begin{cases} \operatorname{tr} A(\log_2 A - \log_2 B), & \operatorname{supp} A \subset \operatorname{supp} B \\ \infty, & \text{otherwise,} \end{cases}$$

which is the Umegaki relative entropy between A and B.

2. Let $q \in]0,1[$ and the functions $f_q(x) = \frac{x^q}{q-1} - 1$ $(x > 0)$ and $f_q(0) = 0$. Then,

$$S_{f_q}(A\|B) = \frac{1 - \operatorname{tr} A^q B^{1-q}}{1-q},$$

which is the quantum Tsallis relative entropy between A and B.

3. If $f(x) = (\sqrt{x} - 1)^2$ $(x \geq 0)$, then $S_f(A\|B) = \left\|\sqrt{A} - \sqrt{B}\right\|_{\mathrm{HS}}^2$, where $\|.\|_{\mathrm{HS}}$ stands for the Hilbert-Schmidt norm.

Preliminaries

Theorem (Lajos Molnár)

Let $\phi : S(H) \to S(H)$ *be a surjective transformation which satisfies*

$$S(\phi(A)\|\phi(B)) = S(A\|B) \qquad (A, B \in S(H)).$$

Then ϕ *is of the form*

$$\phi(A) = UAU^* \quad (A \in S(H))$$

with some unitary or antiunitary operator U *on* H*.*

Preliminaries

Theorem (Lajos Molnár)

Let $\phi : S(H) \to S(H)$ be a surjective transformation which satisfies

$$S(\phi(A)\|\phi(B)) = S(A\|B) \qquad (A, B \in S(H)).$$

Then ϕ is of the form

$$\phi(A) = UAU^* \quad (A \in S(H))$$

with some unitary or antiunitary operator U on H.

- this result holds true also in the case when the surjectivity of transformations is not assumed;
- L. Molnár and G. Nagy have determined the general form of the maps on density operators preserving other kinds of relative entropies.

Relation with Wigner's fundamental theorem

Quantum mechanical symmetry transformations:

- bijective maps ϕ on the space $P_1(H)$;
- $\operatorname{tr} \phi(P)\phi(Q) = \operatorname{tr} PQ \quad (P, Q \in P_1(H))$.

Relation with Wigner's fundamental theorem

Quantum mechanical symmetry transformations:

- bijective maps ϕ on the space $P_1(H)$;
- $\operatorname{tr} \phi(P)\phi(Q) = \operatorname{tr} PQ \quad (P, Q \in P_1(H))$.

- Every map of the form $P \mapsto UPU^*$ $(P \in P_1(H))$ where $U \colon H \to H$ is a unitary or an antiunitary operator is a symmetry transformation;
- **Wigner's theorem**: the reverse statement is also true;
- There is a non-surjective version of the theorem of Wigner.

Main result

Theorem

Assume that f is strictly convex and that $\phi\colon S(H) \to S(H)$ *is a map which satisfies*

$$S_f(\phi(A)||\phi(B)) = S_f(A||B) \quad (A, B \in S(H)).$$

Then there is either a unitary or an antiunitary operator U on H such that ϕ *is of the form*

$$\phi(A) = UAU^* \qquad (A \in S(H)).$$

Main result

Theorem

Assume that f is strictly convex and that $\phi\colon S(H) \to S(H)$ is a map which satisfies

$$S_f(\phi(A)||\phi(B)) = S_f(A||B) \quad (A, B \in S(H)).$$

Then there is either a unitary or an antiunitary operator U on H such that ϕ is of the form

$$\phi(A) = UAU^* \qquad (A \in S(H)).$$

The previously mentioned functions

- $f(x) = x\log_2 x$ $(x > 0)$ and $f(0) = 0$
- $f_q(x) = \frac{x^q}{q-1} - 1$ $(x > 0)$ and $f_q(0) = 0$
- $f(x) = (\sqrt{x} - 1)^2$ $(x \geq 0)$

satisfy the conditions of the theorem.

References

I. Csiszár, *Information type measure of difference of probability distributions and indirect observations*, Studia Sci. Math. Hungar. **2** (1967), 299–318.

F. Hiai, M. Mosonyi, D. Petz and C. Bény, *Quantum f-divergences and error correction*, Rev. Math. Phys. **23** (2011), 691–747.

L. Molnár, *Selected Preserver Problems on Algebraic Structures of Linear Operators and on Function Spaces*, Lecture Notes in Mathematics, Vol. 1895, Springer, 2007.

L. Molnár, *Maps on states preserving the relative entropy*, J. Math. Phys. **49** (2008), 032114.

L. Molnár and G. Nagy, *Isometries and relative entropy preserving maps on density operators*, Linear and Multilinear Algebra, **60** 2012, 93–108.

L. Molnár and P. Szokol, *Maps on states preserving the relative entropy II*, Linear Algebra Appl., **432** (2010), 3343–3350.