Lyapunov stability of the solution for the impulsive differential equations

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Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with norm $\|\cdot\|$ and let $\mathbb{R}_+ = [0, \infty)$.

Let Ω be the phase space of some evolutionary process .

Denote by P_t the point mapping the process at the moment t .

assume that the state of the process is determined by n parameters.

Then P_t as a point (t, x) of the(n+1)–dimension space $\mathbb{R}^{n+1}$ and Ω as a set in $\mathbb{R}^n$.

Let $t_0 \in \mathbb{R}, r = \text{const} > 0, \Omega \subseteq \mathbb{R}^n, \Omega \neq \varnothing$ and

$J \subseteq \mathbb{R}.$

$PC[J,\Omega] = \{\sigma : J \to \Omega\} : \sigma(t)$ is a piecewise continuous function with points of discontinuity $\tilde{t} \in J$ at which $\sigma(\tilde{t} - 0)$ and $\sigma(\tilde{t} + 0)$ and $\sigma(\tilde{t} - 0) = \sigma(\tilde{t})$.

$C(J_k,\mathbb{R})$: set of continuous function on J_k

$C^n(J_k,\mathbb{R})$: set of continuous differentiable function order n on J_k

When $k = 0,1,2,\ldots,m$

Consider

$$\left.\begin{aligned}
&-x''(t) = f\left(t, x(t), x\left(\alpha(t)\right)\right) - M(t)x'(t) \equiv Fx(t),\\
&\qquad\qquad\qquad t \neq t_k,\ t \in J = [-r, 0],\\
&\Delta x(t) = I_k\left(x(t), x'(t)\right),\ t = t_k,\quad k = 1, 2, \ldots, m,\\
&\Delta x'(t) = I_k^*\left(x(t), x'(t)\right),\ t = t_k,\quad k = 1, 2, \ldots, m,\\
&x(0) = x(J),\quad x'(0) = x'(J).
\end{aligned}\right\} \quad (1)$$

When $f \in C^2\left(J \times \mathbb{R}^2, \mathbb{R}\right), \alpha \in C(J, \mathbb{R}), 0 \le \alpha(t) \le t,$

$$I_k, I_k^* \in C^2\left(\mathbb{R}^2, \mathbb{R}\right), M \in C\left(J, \mathbb{R}_+\right),$$

$$\Delta x(t) = x(t+0) - x(t-0), \qquad (2)$$

$$\Delta x'(t) = x'(t+0) - x'(t-0),$$

and for $t \ge t_0, x_t \in PC[J, \Omega]$ is defined by

$$x_t(s) = x_t(t+s), -r \le s \le 0.$$

Let $\varphi_0 \in PC[J,\Omega]$. Denote by $x(t) = x(t;t_0,\varphi_0)$ the solution of system (1), (2), satisfying the initial conditions

$$\begin{cases} x(t;t_0,\varphi_0) = \varphi_0(t-t_0), \quad t_0 - r \le t \le t_0 \\ x(t_0+0;t_0,\varphi_0) = \varphi_0(0) \\ x'(t_0+0;t_0,\varphi_0) = \varphi_0'(0) \end{cases} \qquad (3)$$

Let $J_1 = [t_0,\omega)$, $J_2 = [t_0,\tilde{\omega})$ and $J_1 \subseteq J_2$.

From Stamova et al., 2009

Introduce the following conditions:

H1.1. The function f is continuous in

$$[t_0,\infty)\times PC[J,\Omega],\ J=[-r,0]$$

H1.2. The function f is locally Lipschitz continuous with respect to its second argument in

$$[t_0,\infty)\times PC[J,\Omega]$$

H1.3. There exists a constant $P > 0$ such that

$$\| f(t, x_t) \| \le P < \infty \text{ for } (t, x_t) \in [t_0, \infty) \times PC[J, \Omega]$$

H1.4. The functions τ_k are Lipschitz continuous with respect to $x \in \Omega$ with Lipschitz constants L_k,

$$0 \le L_k < 1/p, k = 1, 2, \ldots$$

H1.5. $t_0 < \tau_1(x) < \tau_2(x) < \ldots, \ x \in \Omega$

H1.6. $\tau_k(x) \to \infty$ as $k \to \infty$, uniformly on $x \in \Omega$

H1.7. $\tau_k(x + I_k(x)) \le \tau_k(x)$ and

$$\tau_k(x + I_k^*(x)) \le \tau_k^*(x) \text{ for } x \in \Omega, k = 1, 2, \ldots$$

H1.8. For each $(t_0, \varphi_0) \in \mathbb{R} \times PC[J, \Omega]$, the solution of initial value problem without impulses (1), (3) does not leave the domain Ω for $t \ge t_0$.

H1.9. $(E + I_k) : \Omega \to \Omega, k = 1, 2, \ldots,$ where E is the identity in Ω.

H1.10. The functions I_k, I_k^* are Lipschitz continuous with respect to $x \in \Omega$ with Lipschitz constants $\Lambda_k, 0 \le \Lambda_k < 1 - L_k P,\ k = 1, 2, \ldots$.

H1.11. $I_k, I_k^* \in C[\Omega, \Omega],\ k = 1, 2, \ldots$.

H1.12. $t_0 < t_1 < t_2 < \ldots < t_k < t_{k+1} < \ldots$.

H1.13. $\lim_{k \to \infty} t_k = \infty$

Let $\tau_0(x) \equiv t_0$ for $x \in \Omega$. Introduced the following condition:

H2.1. $\tau_k \in C[\Omega, (t_0, \infty)]$, $k = 1, 2, \ldots$.

Assuming that H2.1, H1.5, and H1.6 are fulfilled, we consider

$$\sigma_k = \{(t, x) : t = \tau_k(x), x \in \Omega\}, \ k = 1, 2, \ldots.$$

Introduce the following notations:

$\|\varphi\|_k = \sup_{t\in[t_0-r,t_0]} \|\varphi(t-t_0)\|$ is the norm of the function

$$\varphi \in PC[J,\Omega]$$

$K = \{a \in C[\mathbb{R}_+,\mathbb{R}_+] : a(r)$ is strictly increasing

and $a(0)=0\}$.

In the case $r=\infty$ we have

$$\|\varphi\|_r = \|\varphi\|_\infty = \sup_{t\in(-\infty,t_0]} \|\varphi(t-t_0)\|$$

Introduce the following conditions:

H2.2. $f(t,0)=0,\ t\geq t_0.$

H2.3. $I_k(0)=0$ and $I_k^*(0)=0,\ k=1,2,\ldots.$

H2.4. The integral curves of the system (1) meet successively each one of the hypersurfaces

$\sigma_1,\sigma_2,\ldots.$ exactly once.

Let $t_1,t_2,\ldots.(t_0<t_1<t_2<\ldots.)$ be the moments in which the integral curve $(t,x(t;t_0,\varphi_0))$ of problem (1), (2) meets $\sigma_k,\ k=1,2,\ldots.$

Definition 1. The zero solution $x(t) \equiv 0$ of system (1) is said to be:

(a) *stable*, if

$$\left(\forall t_0 \in \mathbb{R}\right)\left(\forall \varepsilon > 0\right)\left(\exists \delta = \delta\left(t_0, \varepsilon\right) > 0\right)$$

$$\left(\forall \varphi_0 \in PC\left[J, \Omega\right] : \|\varphi_0\|_r < \delta\right)\left(\forall t \geq t_0\right) : \left\|x\left(t; t_0, \varphi_0\right)\right\| < \varepsilon;$$

(b) *uniformly stable*, if the number in (a) is independent of $t_0 \in \mathbb{R}$;

(c) *attractive*, if

$$\left(\forall t_0 \in \mathbb{R}\right)\left(\exists \lambda = \lambda\left(t_0\right) > 0\right)\left(\forall \varphi_0 \in PC\left[J,\Omega\right] : \left\|\varphi_0\right\|_r < \lambda\right) :$$

$$\lim_{t\to\infty} x\left(t;t_0,\varphi_0\right) = 0;$$

(d) *equi-attractive*, if

$$\left(\forall t_0 \in \mathbb{R}\right)\left(\exists \lambda = \lambda\left(t_0\right) > 0\right)\left(\forall \varepsilon > 0\right)\left(\exists T = T\left(t_0,\varepsilon\right) > 0\right)$$

$$\left(\forall \varphi_0 \in PC\left[J,\Omega\right] : \left\|\varphi_0\right\|_r < \lambda\right)\left(\forall t \geq t_0 + T\right) : \left\|x\left(t;t_0,\varphi_0\right)\right\| < \varepsilon;$$

(e) *uniformly attractive*, if the numbers λ and T in (d) are independent of $t_0 \in \mathbb{R}$;

(f) *asymptotically stable*, if it is stable and attractive;

(g) *uniformly asymptotically stable*, if it is uniformly stable and uniformly attractive;

(h) *unstable*, if

$$(\exists t_0 \in \mathbb{R})(\exists \varepsilon > 0)(\forall \delta > 0)\left(\exists \varphi_0 \in PC[J,\Omega] : \|\varphi_0\|_r < \delta\right)$$

$$(\exists t \geq t_0) : \|x(t;t_0,\varphi_0)\| \geq \varepsilon;$$

In the proofs of main theorems, we use piecewise continuous Lyapunov functions

$$V : [t_0, \infty) \times \Omega \to \mathbb{R}_+, V \in V_0$$

for which the following condition is true:

H2.5. $V(t, 0) = 0, t \geq t_0.$

Theorem 1. Assume that:

(1) *Conditions* H1.1, H1.2, H1.3, H1.5, H1.6, H1.9, H1.11, H2.1–H2.4 *hold.*

(2) There exists a function $V \in V_0$ such that H2.5 holds,

$$a(\|x\|) \le V(t,x),\ a \in K,\ (t,x) \in [t_0,\infty) \times \Omega \qquad (4)$$

$$V(t+0, x+I_k(x)) \le V(t,x),\ \ (t,x) \in \sigma_k,\ \text{k=1,2,\ldots,} \qquad (5)$$

and the inequality

$$D^+_{(1)}V(t,x(t)) \le 0,\ \ t \ne \tau_k(x(t)),\ \text{k=1,2,\ldots,}$$

is valid for $t \in [t_0,\infty),\ x \in \Omega_1$.

Then the zero solution of system (1) is stable.

Theorem 2. Let the conditions of Theorem 1 hold, and let a function $b \in K$ exist such that

$$V(t,x) \le b(\|x\|), (t,x) \in (t_0,\infty) \times \Omega. \quad (8)$$

Then the zero solution of system (1) is uniformly stable.