Reinhardt domains in spaces of continuous functions

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CLASSICAL REINHARDT DOMAINS

domain: open connected set

D **Reinhardt** domain in $\mathbb{C}^n$:

$$(z_1, \ldots, z_n) \in D \Longrightarrow \big\{(u_1, \ldots, u_n) : \ |u_k| = |z_k|, \ k = 1, \ldots, n\big\} \subset D$$

D **complete** Reinhardt dom. in $\mathbb{C}^N$:

$$(z_1, \ldots, z_n) \in D \Longrightarrow \big\{(u_1, \ldots, u_n) : \ |u_k| \le |z_k|, \ k = 1, \ldots, n\big\} \subset D$$

Sunada (1974):

$D, \widetilde{D}$ holomorphically equiv. bded compl. R-domains in $\mathbb{C}^N$ $\Longrightarrow$

$$\exists\ L \in \mathcal{L}(\mathbb{C}^n) \qquad LD_+ = \widetilde{D}_+$$

GENERALIZED CONCEPT

$(E, \mathrm{Re}, \leq)$ complex vector lattice:

$$|x| = \max\left\{\mathrm{Re}(e^{it}x) : \ t \in \mathbb{R}\right\} \quad \text{well-defined}$$

Reinhardt domain: $D = \{x \in E : \ |x| \in D\}$

complete Reinhardt domain: $D = \{x \in E : \ \exists\, y \in D_+ \ \ |x| \leq y\}$

Barton-Dineen-Timoney (1983):

R-dom.s in separable Banach space with basis

Vigué (1998):

$E = \mathcal{C}(\Omega)$, Ω compact, $r : \Omega \to [1, M]$ lower semicontinuous

$$D_r := \left\{ f \in E : \ |f(\omega)| < r(\omega), \ \omega \in \Omega \right\}$$

Then D *is symmetric* $\iff$ r *is continuous,*

In general $\mathrm{Aut}(D)0 = \left\{ f \in E : \ r(\omega) \neq \lim_{\eta \to \omega} r(\eta) \Rightarrow f(\omega) = 0 \right\}$

Definition *(Stachó-Zalar, 2003):*

Continuous R-dom. (CRD)$\equiv$ *complete R-dom. in some* $\mathcal{C}_0(\Omega)$

COUNTER-EXAMPLE TO SUNADA-TYPE THEOREMS

Stachó (2007):
Construction of **lin. equiv.** *CRDs* $D \subset \mathcal{C}(\Omega)$ *and* $\widetilde{D} \subset \mathcal{C}(\widetilde{\Omega})$ *with*

$$\nexists \text{ linear isomorphism } L : D \leftrightarrow \widetilde{D} \qquad L\Big(\mathrm{Re}\,\mathcal{C}(\Omega)\Big) = \mathrm{Re}\,\mathcal{C}(\widetilde{\Omega})$$

$$\Omega = \{\pm 1\} \times \mathbb{T}\ , \qquad \widetilde{\Omega} = \{(e^{it/2}, e^{it}) :\ t \in \mathbb{R}\}$$

border of a *cylindric* resp. *Möbius* band with middle circle $\{0\} \times \mathbb{T}$.

$\omega_{k,t} := \big((-1)^k, e^{it}\big), \quad \widetilde{\omega}_{k,t} := \big((-1)^k e^{it/2}, e^{it}\big)$

$$D := \{f \in \mathcal{C}(\Omega) :\ |f(\omega_{1,t})|^2 + |f(\omega_{2,t})|^2 < 1,\ 0 \le t < 2\pi\}$$
$$\widetilde{D} := \{\widetilde{f} \in \mathcal{C}(\widetilde{\Omega}) :\ |\widetilde{f}(\widetilde{\omega}_{1,t})|^2 + |\widetilde{f}(\widetilde{\omega}_{2,t})|^2 < 1,\ 0 \le t < 2\pi\}$$

SYMMETRIC CASE

Stachó-Zalar (2003): $D \subset E := \mathcal{C}_0(\Omega)$ **bded. symm.** *CRD* $\Longrightarrow$

$$D = \Big\{ f : \sum_{\omega \in \Omega_j} m(\omega) |f(\omega)|^2 < 1, \ j \in J \Big\}$$

where $\bigcup^{\bullet}_{j \in J} \Omega_j = \Omega, \ \ m : \Omega \to \mathbb{R}_+$ *such that*

$$\sup_{j \in J} \# \Omega_j < \infty, \quad 0 < \inf m \le \sup m < \infty.$$

Proof with **Jordan theory**.

HOLOMORPHIC HULL

D domain in E Banach space
$\exists\, \mathcal{D}$ max Riemann surface $\supset D$ with hol.ext. $\mathrm{Hol}(D) \hookrightarrow \mathrm{Hol}(\mathcal{D})$
$[\mathcal{D}$ manifold, $\exists\, \Phi : \mathcal{D} \to E$ charts $= \{\Phi|U : U \in \mathbf{U}\}$, $\Phi|D = \mathrm{id}]$
$\mathcal{D}$ unique up to hol. equiv.

Kaup 1970: *D circular $(0 \in D = \mathbb{T}D) \Longrightarrow \mathcal{D}$ embeds into E*
$\exists!\, \widehat{D} \subset E$ *max.domain with* hol.ext. $\mathrm{Hol}(D) \hookrightarrow \mathrm{Hol}(\widehat{D})$

Remark: $D \subset \mathbb{C}^N$ classical Reinhardt-domain $\Longrightarrow$
$\widehat{D} =$ log-conv hull of D.

Question (R. Szőke, 2012):
Is there a reasonable concept of **logarithmic convexity** *for CRDs?*

LOG-CONVEX R-DOMAINS IN FUNCTION SPACES

$\mathcal{F} \subset \{\Omega \to \mathbb{C} \text{ functions}\}$ complex lattice ideal

$$f \in \mathcal{F},\ |g| \leq |f| \Rightarrow g \in \mathcal{F}$$

Definition. $D \subset \mathcal{F}$ is an **R-domain**:

$$f \in D \Rightarrow \{g \in \mathcal{F} : |g| = |f|\} \subset D$$

The R-domain D is **complete** if

$$f \in D \Rightarrow \{g \in \mathcal{F} : |g| = |f|\} \subset D$$

The complete R-domain D is **log-convex** if

$$f_1, f_2 \in D,\ 0 \leq \lambda \leq 1 \Rightarrow \{g \in \mathcal{F} :\ |g| \leq |f_1|^{\lambda} |f_2|^{1-\lambda}\} \subset D$$

EXTENSION TO THE LOG-CONVEX HULL

Definition. The *log-convex hull* of an R-domain is the intersection of all log-convex R-domains containing it.

Remark. $\big[\text{log-conv hull of } D\big] =$
$$\bigcup_{n=2}^{\infty} \Big\{ g \in \mathcal{F} : \exists\, f_1, \ldots, f_n \in D \ \exists\, \lambda_1, \ldots, \lambda_n \geq 0 \ \sum_{k=1}^{n} \lambda_k = 1, \ |g| \leq \prod_{k=1}^{n} |f_k|^{\lambda_k} \Big\}$$

Theorem.
If D is a complete R-domain in $E := \mathcal{C}_0(\Omega)$ then every holomorphic function $D \to \mathbb{C}$ extends holomorphically to the log-convex hull $\widehat{D}$ of D.

Conjecture. *If $\dim(E) > 1$ then the holomorphic hull of any CRD $D \subset E$ is a complete CRD.*

D CRD in $E := \mathcal{C}_0(\Omega)$, $\widehat{D}$ the hol.hull regarded as a circ.dom. in E

(1) *Let* $H := \{ \sum_k \zeta_k u_N : (\zeta_1, \dots, \zeta_N) \in \Delta \}$,
$u_1, \dots, u_N \in E$ *lin.indep.*, $\Delta \subset \mathbb{C}^N$ *complete R-dom.*
Assume $H + U \subset D$, $U \subset E$ *convex 0-neighbh.* $\Longrightarrow$
$\widehat{D} \supset \widetilde{H} + U$, $\widetilde{H} := \{ \sum_k \zeta_k u_k : (\zeta_1, \dots, \zeta_N) \in \widetilde{\Delta} \}$, $\widetilde{\Delta} :=$ *log-conv.h.*(Δ).

(2*) $f \in D \Longrightarrow \exists \varepsilon > 0 \ \forall h \in E \ |h| \le |f| + \varepsilon \Rightarrow h \in D$
Remark. Trivial, but the *converse needs* $\widehat{D}$ *to be* **complete** *R-dom.*

(3) $f \in E_0$, $\varepsilon > 0$, $\{u_1, \dots, u_N\} \subset E_+$ *part. of unity over* $\mathrm{support}(f)$
$\mathrm{diam} f(\mathrm{support}(u_k)) \le \varepsilon \ (\forall k)$, $1 \ge \sum_k u_k \ge 1_{\mathrm{support}(f)}$.
Assume $\omega_k \in \mathrm{support}(u_k)$ $(\forall k)$. *Then* $\left\| f - \sum_k f(\omega_k) u_k \right\| \le \varepsilon$.

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INGREDIENTS OF PROOF

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