ON THE EXISTENCE OF A UNIFORM EXPONENTIAL DICHOTOMY OF EVOLUTION FOR A STRONGLY CONTINUOUS EVOLUTION FAMILY

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Outline

Origin of the problem

Non-autonomous Cauchy problem

We consider the non-autonomous abstract Cauchy problem

$$\dot{x}(t) = A(t)x(t), \quad x(s) = x_s \in D(A(s)),\ t \geq s,\ t,s \in J, \tag{1}$$

on a Banach space X, where $J = \mathbb{R}$ or $J = \mathbb{R}_+$. We assume that the above Cauchy problem is well-posed, i.e. there exists an exponentially bounded, strongly continuous evolution family $\{\Phi(t,s)\}_{t\geq s}$ (see Definition 1), such that $x(\cdot)$ given by

$$x(t) = \Phi(t,s)x_s,\ t \geq s,$$

is differentiable for any given initial conditions $x(s) = x_s \in D(A(s)),\ x(t) \in D(A(t))$ and (1) holds.

Origin of the problem

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Mild solution

Now let f be a locally integrable X-valued function on J and consider the inhomogeneous equation

$$\dot{x}(t) = A(t)x(t) + f(t), \quad t \in J. \tag{2}$$

A function $u(\cdot)$ is called a *mild* solution of (2) if

$$u(t) = \Phi(t,s)u(s) + \int_t^s \Phi(t,\tau)f(\tau)d\tau, \qquad t \geq s,\ t,s \in J, \tag{3}$$

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Origin of the problem

Definition 1

A family of bounded linear operators acting on a Banach space X denoted by $\Phi = \{\Phi(t,s)\}_{t \geq s \geq 0}$ is called an *uniform exponentially bounded, strongly continuous evolution family* if the following properties hold:

(i) $\Phi(t,t) = I$ (where I is the identity on X), for all $t \geq 0$;

(ii) $(t,s,x) \mapsto \Phi(t,s)x$ is continuous for every $t \geq s \geq 0$ and $x \in X$;

(iii) $\Phi(t,s) = \Phi(t,\tau)\Phi(\tau,s)$, for all $t \geq \tau \geq s \geq 0$;

(iv) there exist two constants $M, \omega > 0$ such that

$$\|\Phi(t,s)\| \leq Me^{\omega(t-s)}, \text{ for all } t \geq s \geq 0.$$

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Origin of the problem

History of the problem

If $\{A(t)\}_{t\geq 0}$ is a family of linear and bounded operators that belongs to M^1 (i.e. $\sup_{t\geq 0}\int_t^{t+1}\|A(\tau)\|d\tau < \infty$), then the above Cauchy problem (1) is well-posed (and moreover $\Phi(t,s)$ is an invertible operator, for each $t \geq s \geq 0$.

Some other cases when the problem (1) is well-posed, for certain unbounded operator family $\{A(t)\}_{t\geq 0}$, were pointed out by Chicone and Latushkin [12], Curtain and Prichard [13], Pazy [29], Levitan and Zhikov [21] or Tanabe [40].

The characterization of the uniform exponential dichotomy of (1) in terms of the solution of (3) has a fairly long history and goes back to the work of O. Perron [30] in 1930. His result served as a starting point for many works on the qualitative theory of the solutions of differential equations.

Origin of the problem

History of the problem

Relevant results concerning the extension of Perron's problem in the more general framework of the infinite-dimensional Banach spaces (but with $A(t)$ bounded) were obtained by J.L.Daleckij and M.G.Krein [14]. J.L. Massera and J.J. Schäffer [22]. For certain cases of unbounded $A(t)$ we refer the reader to the work of Levitan and Zhikov (see [21]).

More recently another approach uses frequently the so-called *evolution semigroup* on some suitable space of X-valued functions induced by the evolution family (see [41], [42], [35], [36], [12], [19], [37], [38], [18], [20], [43]). It is worth to note that most of the results in this direction are restricted to the case where $J = \mathbb{R}$. The case of the positive semiaxis has been considered in [12] and [43].

Origin of the problem

History of the problem

Over the last decades it can be seen an increasing interest in the study of the asymptotic behavior of evolution equations in abstract spaces. We refer the reader to [1], [2], [17], [23], [25], [16], [31], [32], [33], [34], [39], [24]. Important contributions in the study of the existence of an exponential dichotomy for evolution equations has been made and it is worth to note here few works by Nguyen Thieu Huy (see [25], [26], [27], [28]) and Luis Barreira and Claudia Valls (see [3], [4], [5], [6], [7], [8], [9], [10]).

Exponential dichotomy

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Let X be a Banach space and $B(X)$ the Banach algebra of all linear and bounded operators acting on X (the norms on both X and $B(X)$ will be denoted by $\|\cdot\|$).

Definition 2

Let $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ be a uniform exponentially bounded, strongly continuous evolution family. We say that $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ admits a **uniform exponential dichotomy** if there exist a family of projectors $\{P(t)\}_{t\geq 0}$ and $N,\nu>0$ such that:

- $\Phi(t,t_0)P(t_0)=P(t)\Phi(t,t_0)$, for all $t\geq t_0\geq 0$:
- $\Phi(t,t_0): KerP(t_0)\to KerP(t)$ is invertible:
- $\|\Phi(t,t_0)P(t_0)x\|\leq Ne^{\nu(t-t_0)}\|P(t_0)x\|$, for all $t\geq t_0\geq 0$, and $x\in X$:
- $\|\Phi(t,t_0)(I-P(t_0))x\|\geq Ne^{\nu(t-t_0)}\|(I-P(t_0))x\|$, for all $t\geq t_0\geq 0$, and $x\in X$

Exponential dichotomy

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L^p spaces

As usual, we denote by $L^p(\mathbb{R}_+, X)$ $(p \geq 1)$ the space of equivalence classes of Bochner measurable functions $f : \mathbb{R}_+ \to X$, such that

$$\int_0^\infty \|f(t)\|^p < \infty.$$

As known, $L^p(\mathbb{R}_+, X)$ endowed with the norm

$$\|f\|_p = \left(\int_0^\infty \|f(t)\|^p \right)^{\frac{1}{p}}$$

is a Banach space.

Exponential dichotomy

Splitting

We set now $X_1^q(0) = \{x \in X : \Phi(\cdot, 0)x \in L^q(\mathbb{R}_+, X)\}$. *Throughout this paper, we will assume that* $X_1^q(0)$ *is closed and that there exists a closed subspace* $X_2^q(0)$ *such that* $X = X_1^q(0) \oplus X_2^q(0)$.

Also, we will denote by $\{P_1^q(0)\}$ and $\{P_2^q(0)\}$ the corresponding families of projectors, i.e. $ImP_1^q(0) = X_1^q(0)$, $ImP_2^q(0) = X_2^q(0)$. It is known that $P_i^q(0) \in B(X)$ and $(P_i^q(0))^2 = P_i^q(0)$, $i = 1, 2$.

Admissibility

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Definition 3

Let $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ be an uniform exponentially bounded, strongly continuous evolution family. The pair $(L^p(\mathbb{R}_+,X), L^q(\mathbb{R}_+,X))$ is said to be *admissible to* $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ if for each $f\in L^p(\mathbb{R}_+,X)$, there exists $x\in X$ such that

$$x_f(\cdot):\mathbb{R}_+\to X, \qquad \text{given by } x_f(t)=\Phi(t,0)x+\int_0^t \Phi(t,\tau)f(\tau)d\tau$$

belongs to $L^q(\mathbb{R}_+,X)$.

Admissibility

Remark

If $x \in X_2^q(0)$, $x \neq 0$, then $\Phi(t,0)x \neq 0$, for each $t \geq 0$.

Proposition 1

Let $\{\Phi(t,t_0)\}_{t \geq t_0 \geq 0}$ be an uniform exponentially bounded, strongly continuous evolution family. If the pair $(L^p(\mathbb{R}_+,X), L^q(\mathbb{R}_+,X))$ is admissible to $\{\Phi(t,t_0)\}_{t \geq t_0 \geq 0}$ then for every $f \in L^p(\mathbb{R}_+,X)$ there is an unique $x \in X_2^q(0)$ such that

$$x_f(\cdot) : \mathbb{R}_+ \to X, \qquad x_f(t) = \Phi(t,0)x + \int_0^t \Phi(t,\tau)f(\tau)d\tau$$

belongs to $L^q(\mathbb{R}_+,X)$.

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Boundedness lemma

Lemma 1

Let $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ be an uniform exponentially bounded, strongly continuous evolution family. If the pair $(L^p(\mathbb{R}_+,X),L^q(\mathbb{R}_+,X))$ is admissible to $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ then there is $K>0$ such that

$$\|x_f\|_q \leq K\|f\|_p \text{ and } \|x_f(0)\| \leq K\|f\|_p,$$

for all $f\in L^p(\mathbb{R}_+,X)$, with $x_f(0)\in X_2^q(0)$.

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Boundedness lemma

Lemma 2

Let $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ be an uniform exponentially bounded, strongly continuous evolution family. If the pair $(L^p(\mathbb{R}_+,X),L^q(\mathbb{R}_+,X))$ is admissible to $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$, then there exists $L>0$ such that

$$\|x_f(t)\| \leq e^{\omega}(KL+1)\|f\|_p, \quad \text{for all } t\geq 0.$$

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Main results

Theorem 1

If $(L^p(\mathbb{R}_+,X),L^q(\mathbb{R}_+,X))$ is admissible to $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$, then $(L^p(\mathbb{R}_+,X),L^\infty(\mathbb{R}_+,X))$ is admissible to $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$.

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Main results

Theorem 2

Let $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ be an uniform exponentially bounded, strongly continuous evolution family. If the pair $(L^p(\mathbb{R}_+,X),L^q(\mathbb{R}_+,X))$ is admissible to $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$, $(p,q)\neq(1,\infty)$ and $X_1^q(t_0)=\{x\in X:\Phi(\cdot,t_0)x\in L^q(\mathbb{R}_+,X)\}$, then:

- $X_1^q(t_0)$ is a complemented subspace for all $t_0\geq 0$;
- $X_2^q(t_0)=\Phi(t_0,0)X_2^q(0)$ is a complement of $X_1^q(t_0)$, for all $t_0\geq 0$;
- $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ is exponentially dichotomic;
- $X_1^q(t_0)=X_1(t_0)$, for all $t_0\geq 0$;

Main results

Theorem 2 (cont.)

- the family of projectors $\{P_1^q(t_0)\}_{t_0 \geq 0}$ associated to the decomposition $X = X_1^q(t_0) \oplus X_2^q(t_0)$ has the following property:

$$t \mapsto P_1^q(t)x : \mathbb{R}_+ \to X \text{ is continous for all } \ x \in X$$

and

$$\sup_{t \geq 0} \|P_1^q(t)\| < \infty.$$

- A. Ben-Artzi, I. Gohberg; *Dichotomies of systems and invertibility of linear ordinary differential operators*, Oper. Theory Adv. Appl. 56(1992), 1-42.
- A. Ben-Artzi, I. Gohberg, M.A. Kaashoek; *Invertibility and dichotomy of differential operators on the half-line*, J. Dynam. Differential Equations 5(1993), 1-36.
- L. Barreira, Ya. Pesin; *Lyapunov Exponents and Smooth Ergodic Theory*, Univ. Lecture Ser., vol. 23, Amer. Math. Soc., 2002.
- L. Barreira, Ya. Pesin; *Nonuniform Hyperbolicity*, Encyclopedia Math. Appl., vol. 115, Cambridge University Press, 2007.
- L. Barreira, J. Schmeling; *Sets of "non-typical" points have full topological entropy and full Hausdorff dimension*, Israel J. Math. 116(2000), 29-70.

- L. Barreira, C. Valls; *Optimal estimates along stable manifolds of non-uniformly hyperbolic dynamics*, Proc. Roy. Soc. Edinburg Sect. A 138(2008), 693-717.
- L. Barreira, C. Valls; *Nonuniform exponential dichotomies and Lyapunov regularity*, Journal of Dynamics and Differential Equations 19(2007), 215-241.
- L. Barreira, C. Valls; *Stability of Nonautonomous Differential Equations*, Lecture Notes in Math., vol. 1926, Springer, 2008.
- L. Barreira, C. Valls; *Nonuniform exponential contractions and Lyapunov sequences*, Journal of Differential Equations 246(2009), 4743-4771.
- L. Barreira, C. Valls; *Admissibility for nonuniform exponential contractions*, Journal of Differential Equations 249(2010), 2889-2904.

- L. Barreira, C. Valls *Nonuniform exponential dichotomies and admissibility*, Discrete and Continuous Dynamical Systems, 30, no. 1, (2011), 39 - 53.
- C. Chicone, Y. Latushkin; *Evolution semigroups in dynamical systems and differential equations*, Mathematical Surveys and Monographs, vol. 70, Providence, RO: American Mathematical Society, 1999.
- R. Curtain, A.J. Prichard; *Infinite-Dimensional Linear Systems Theory*, Lecture Notes in Control and Inform. Sci., vol. 8, Springer, New York, 1978.
- J.L. Daleckij, M.G. Krein; *Stability of Differential Equations in Banach Space*, Amer. Math. Soc., Providence, RI, 1974.
- P. Hartman; *Ordinary Differential Equations*, Wiley, New York/London/Sidney, 1964.

- M. A. Kaashoek and S.M. Verduyn Lunel; *An integrability condition on the resolvent for hyperbolicity of the semigroup*, J. Diff. Eqs. **112** (1994), 374-406 .
- K.J. Engel, R. Nagel; *One - Parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics, 194, Springer, 1999.
- Y. Latushkin, S. Montgomery-Smith; *Evolutionary semigroups and Lyapunov theorems in Banach spaces*, J. Funct. Anal. 127 (1995), 173ï¿½197.
- Y. Latushkin, T. Randolph; *Dichotomy of differential equations on Banach spaces and an algebra of weighted composition operators*, Integral Equations Operator Theory 23 (1995), 472-500.
- Y. Latushkin, T. Randolph, R. Schnaubelt; *Exponential dichotomy and mild solutions of non-autonomous equations*

in Banach spaces, J. Dynam. Differential Equations 3 (1998), 489-510.

- B.M. Levitan, V.V. Zhikov; *Almost Periodic Functions and Differential Equations*, Cambridge Univ. Press, Cambridge, 1982.
- J.L. Massera, J.J. Schäffer; *Linear Differential Equations and Function Spaces*, Academic Press, New York, 1966.
- Nguyen Thieu Huy, Nguyen Van Minh; *Exponential dichotomy of difference equations and applications to evolution equations on the half-line*, Computers and Math. with Appl. 42(2001), 301-311.
- Nguyen Thieu Huy, Nguyen Van Minh; *Characterizations of dichotomies of evolution equations on the half-line*, J. Math. Anal. Appl. 261(2001), 28-44.
- Nguyen Thieu Huy; *Exponentially dichotomous operators and exponential dichotomy of evolution operators on the*

half-line, Integral Equations Operator Theory 48(2004), 497-510.

- Nguyen Thieu-Huy; *Exponential dichotomy of evolution equations and admissibility of function spaces on a half-line*, J. Funct. Anal. 235(2006), 330-354.
- Nguyen Thieu-Huy; *Stable manifolds for semi-linear evolution equations and admissibility of function spaces on a half-line*, J. Math. Anal. Appl. 354(2009), 372-386.
- Nguyen Thieu-Huy; *Invariant manifolds of admissibile classes for semi-linear evolution equations*, J. Differential Equations 246(2009), 1820-1844.
- A. Pazy; *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.
- O. Perron; *Die Stabilitätsfrage bei Differentialgleichungen*, Math. Z. 32(1930) 703-728.

- P. Preda, M. Megan; *Nonuniform dichotomy of evolutionary processes in Banach spaces*, Bull. Austral. Math. Soc. 27(1983), 31-52.
- P. Preda, A. Pogan, C. Preda; (L^p, L^q)*-Admissibility and exponential dichotomy of evolutionary processes on the half-line*, Integr. Equ. Oper. Theory, 49 (3)(2004), 405-418.
- P. Preda, A. Pogan, C. Preda; *Schäffer spaces and uniform exponential stability of linear skew-product semiflows*, J. Differential Equations 212 (2005), 191-207.
- P. Preda, A. Pogan, C. Preda; *Schäffer spaces and exponential dichotomy for evolutionary processes*, J. Differential Equations 230 (2006), 378-391.
- F. Räbiger, R. Schnaubelt; *The spectral mapping theorem for evolution semigroups on spaces of vector valued functions*, Semigroup Forum 48 (1996), 225-239.

- F. Räbiger, R. Schnaubelt; *Absorption evolution families with applications to nonautonomous diffusion processes*, Tubingen Berichte zur Functionalanalysis 5 (1995/1996) 334-335.
- R. Rau; *Hyperbolic evolution groups and dichotomic of evolution families*, J. Dynam. Differential Equations 6 (1994), 107-118.
- R. Schnaubelt; *Sufficient conditions for exponential stability and dichotomy of evolution equations*, Forum Mat. 11 (1999), 543-566.
- R. Schnaubelt; *Asymptotically autonomous parabolic evolution equations*, J. Evol. Eq. 1 (2001), 19-37.
- H. Tanabe; *Equations of Evolution*, Pitman, London, 1979.

- N. van Minh; *Semigroups and stability of nonautonomous differential systems in Banach spaces*, Trans. Amer. Math. Soc. 345 (1994), 223-242.
- N. van Minh; *On the proof of characterization of the exponential dichotomy*, Proc. Amer. Math. Soc. 127 (1999) 779-782.
- N. van Minh, F. Räbiger, R. Schnaubelt; *Exponential stability, exponential expansiveness and exponential dichotomy of evolution equations on the half-line*, Integral Equations Operator Theory 32(1998), 332-353.

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