Lyapunov operator inequalities for exponential stability of linear skew-product semiflows in Banach spaces

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International Workshop on Functional Analysis
Timișoara, Romania, October 12-14, 2012

Contents

Introduction and preliminaries

- The theorem of A. M. Lyapunov shows that if A is a $n \times n$ complex matrix then A has all its characteristics roots with real parts negative if and only if for any positive definite Hermitian matrix H there exists a positive definite Hermitian matrix W satisfying the equation (L_H) $A^*W + WA = -H$ (where $*$ denotes the conjugate transpose of a matrix) (see [2]).

- The use of the Lyapunov equation is extended on the infinte-dimensional framework by Daleckij and Krein ([5]) by replacing the matrix A with a linear and bounded operator acting on a Hilbert space.

- The authors prove that the semigroup $T(t) = e^{tA}$, with $A \in \mathcal{B}(X)$ is exponentially stable if and only if there exists $W \in \mathcal{B}(X)$, $W >> 0$ (i.e. there exists $m > 0$ such that $< Wx, x > \geq m\|x\|^2$, for any $x \in X$), solution of the Lyapunov equation

$$(L) \qquad A^*W + WA = -I.$$

- Moreover, in the same paper it is shown that if W is only *self-adjoint* and it verifies the equation (L), then $\sigma(A) \cap i\mathbb{R} = \emptyset$, which means that the semigroup $T(t)$ is exponentially dichotomic.

- This result is extended by R. Datko in 1970 in [6], for the general case of C_0-semigroups as it follows.

- **Theorem 1.1.** ([6] R. Datko, 1970) A C_0-semigroup $\{T(t)\}_{t\geq 0}$ is exponentially stable if and only if there exists $W \in \mathcal{B}(X)$, $W^* = W$, $W \geq 0$ such that

$$\langle Ax, Wx \rangle + \langle Wx, Ax \rangle = -\|x\|^2 \qquad (1.1)$$

for all $x \in D(A)$, where A denotes the infinitezimal generator of $\{T(t)\}_{t\geq 0}$.

- On the other hand, in 1978, W.A. Coppel ([4]) shows that in finite dimensional spaces the differential system $(A)\quad \dot{x}(t) = A(t)x(t)$ is exponentially dichotomic if and only if there exists $W : \mathbb{R}_+ \to \mathcal{B}(X)$ bounded, with the property that $W^*(t) = W(t)$, for all $t \geq 0$, that satisfies the Lyapunov equation

$$\dot{W}(t) + A^*(t)W(t) + W(t)A(t) = -I \qquad (1.2)$$

  for the nonautonomous case.

- In the proof of this result is essential the fact that the space is finite dimensional, as it is used the fact that the unit ball is dense.

- C. Chicone and Y. Latushkin [3], A. Pazy [11], J. Goldstein [7] and L. Pandolfi [10] studied the Lyapunov operator equations with unbounded A. All the above results are given in the setting of one-parameter semigroups acting on Hilbert spaces.

- In the general case of differential systems in infinite dimensional Hilbert spaces, a similar result can be found in the book of J.L. Massera and J.J. Schäffer ([8]) in the ninth chapter. There is imposed the condition that the complement of the space X_1 is finite dimensional, where that complement will be denoted by X_2 and it is the space which induces dichotomy for the differential system (A). (i.e. X_1 is of finite codimension, as it uses the density of the unit ball in X_2).

- Moreover, in 1998, N.U. Ahmed extends the result of R. Datko from 1970 from Hilbert spaces to Banach spaces for the stability.

- In 2009, in a paper from Systems and Control Letters, C. Preda and P. Preda ([13]) extend the result of R. Datko from 1970 from the stability to the dichotomy of a C_0-semigroup in Hilbert spaces, without the hypothesis of finite codimension of the subspace X_1.

- By using the idea of N.U. Ahmed, in [14] C. Preda and P. Preda study the case of the Lyapunov operator equation for the exponential stability of one-parameter semigroups acting on Banach spaces.

- Also, by using the same techinques as C. Preda and P. Preda, I obtained some results for the case of Lyapunov operator equation for the exponential dichotomy of C_0-semigroups acting on Banach spaces and also for the exponential stability and instability of evolution families acting on Banach spaces.

- For the case of linear skew-product semiflows on real Hilbert spaces, a result which presents an equality of Lyapunov type can be found in the paper of Pham Viet Hai and Le Ngoc Thanh ([15]), who present some characterizations for the uniform exponential stability of linear skew-product semiflows, using a variant of Lyapunov equality.
- In the present paper, we try to go more general and find variants of Lyapunov operator equation for the exponential stability of linear skew-product semiflows acting on Banach spaces.
- This paper extends for the case of linear skew-product semiflows the results obtained in [14] for the case of strongly continuous, one-parameter semigroups acting on Banach spaces, by using analogous techniques.
- In order to do that, we need to recall some notions about the adjoint of a linear operator on a Banach space.

- Let X be a real or complex Banach space and X' its (dual) conjugate space consisting of all bounded and antilinear functionals on X. Also X^* will denote the classic dual space of all bounded and linear functionals on X.

- If Y is also a Banach space we will denote by $\mathcal{B}(X, Y)$ the Banach algebra of all linear and bounded operators from X into Y. If $X = Y$ we will simply write $\mathcal{B}(X)$. The norms on X, X', Y and $\mathcal{B}(X, Y)$ shall be denoted by the symbol $||\cdot||$.

  Definition 1.1. Let X, Y be two Banach spaces and $A \in \mathcal{B}(X, Y)$. Then there exists an unique operator $A^* \in \mathcal{B}(Y', X')$ that satisfies $y(Ax) = A^*y(x)$, for all $x \in X$, $y \in Y'$. A^* will be called *the adjoint of A.*

- It can be easily checked that

  $\|A\| = \|A^*\|$
  $(A+B)^* = A^* + B^*$
  $(\lambda A)^* = \overline{\lambda} A^*$
  If X, Y are reflexive, then $A^{**} = A$.

- It is worth to note that the above notion of the adjoint of a linear and bounded operator between two Banach spaces allows us to create a definition of the adjoint that directly generalizes the definition of the adjoint of an operator on Hilbert spaces.
- In other words, if X and Y are Hilbert spaces and $A \in \mathcal{B}(X, Y)$ then there is no difference of the adjoint between the adjoint A^* defined by considering X, Y to be Hilbert spaces, and the adjoint A^* defined by considering X, Y to be Banach spaces. If we would choose that $A^* : Y^* \to X^*$ then we would obtain a different definition compared to the Hilbert space definition.
- For defining now the concept of a self-adjoint operator on a Banach space we recall that X is isomorphic and isometric with a subspace of X''.

Definition 1.2.

(i) An operator $A \in \mathcal{B}(X, X')$ is said to be *self-adjoint* if the restriction of A^* to X is A, and therefore

$$Ay(x) = \overline{Ax(y)}, \quad \text{for all } x, y \in X. \qquad (1.3)$$

(ii) $A \in \mathcal{B}(X, X')$ is said to be *positive* if A is self-adjoint, and $Ax(x) \geq 0$, for all $x \in X$.

Remark 1.1. It is easy to see that $A \in \mathcal{B}(X, X')$ is positive if and only if $Ax(x)$ is a positive real number, for all $x \in X$.

In the following we will denote by

$$\mathcal{B}^+(X, X') = \{A \in \mathcal{B}(X, X') : A \text{ is positive}\}$$

We will present in what follows some definitions.

Let Θ be a metric space.

Definition 1.3. A map $\sigma : \Theta \times \mathbb{R}_+ \to \Theta$ is said to be a *continuous semiflow on* Θ if the following conditions hold:

i) $\sigma(\theta, 0) = \theta$, for all $\theta \in \Theta$;

ii) $\sigma(\theta, t+s) = \sigma(\sigma(\theta, s), t)$, for all $t, s \in \mathbb{R}_+$ and $\theta \in \Theta$;

iii) $(\theta, t) \mapsto \sigma(\theta, t)$ is continuous on $\Theta \times \mathbb{R}_+$.

If (iii) holds for any $t, s \in \mathbb{R}$ then σ is said to be a *flow on* Θ.

Definition 1.4. Let σ be a continuous semiflow on Θ. A *strongly continuous cocycle over the continuous semiflow* σ is an operator-valued function

$$\Phi : \Theta \times \mathbb{R}_+ \to \mathcal{B}(X), \qquad (\theta, t) \mapsto \Phi(\theta, t),$$

that satisfies the following properties

i) $\Phi(\theta, 0) = I$ (I - the identity operator on X), for all $\theta \in \Theta$;

ii) $(\theta, t) \mapsto \Phi(\theta, t)x$ is continuous for each $\theta \in \Theta$ and $x \in X$;

iii) $\Phi(\theta, t + s) = \Phi(\sigma(\theta, t), s)\Phi(\theta, t)$, for all $t, s \in \mathbb{R}_+$ and $\theta \in \Theta$ (the cocycle identity);

If, in addition,

iv) there exist constants $M, \omega > 0$ such that:

$$||\Phi(\theta, t)|| \leq Me^{\omega t}, \text{ for } t \geq 0 \text{ and } \theta \in \Theta,$$

then the strongly continuous cocycle is *exponentially bounded*.

Definition 1.5. The linear skew-product semiflow (LSPS) associated with the above cocycle is the dynamical system $\pi = (\Phi, \sigma)$ on $\varepsilon = X \times \Theta$ defined by

$$\pi : X \times \Theta \times \mathbb{R}_+ \to X \times \Theta, \ (x, \theta, t) \mapsto \pi(x, \theta, t) = (\Phi(\theta, t)x, \sigma(\theta, t)). \tag{1.4}$$

Definition 1.6. A linear skew-product semiflow (LSPS) $\pi = (\Phi, \sigma)$ on a Banach bundle $\varepsilon = X \times \Theta$ is said to be *exponentially stable* if there exist constants $N, \nu > 0$ such that

$$\|\Phi(\theta, t)x\| \leq Ne^{-\nu t}\|x\|, \text{ for all } t \geq 0, \ \theta \in \Theta, \ x \in X. \tag{1.5}$$

Sufficient condition

In what follows it will be presented a sufficient condition for the exponential stability of linear skew-product semiflows acting on Banach spaces, in terms of Lyapunov inequation.

Theorem 2.1. Let $\pi = (\Phi, \sigma)$ be a linear skew-product semiflow (LSPS). If there exists $W : \Theta \to \mathcal{B}^+(X, X')$ bounded such that

$$W(\sigma(\theta, t))\Phi(\theta, t)x(\Phi(\theta, t)x) + \int_0^t \|\Phi(\theta, \tau)x\|^2 d\tau \leq W(\theta)x(x), \qquad (2.1)$$

for all $t \geq 0$, $\theta \in \Theta$ and $x \in X$, then $\pi = (\Phi, \sigma)$ is exponentially stable.

Proof.
Let $x \in X$, $\theta \in \Theta$ and $t \geq 0$. From (2.1) we have that

$$\int_0^t \|\Phi(\theta,\tau)x\|^2 d\tau \leq K\|x\|^2,$$

for all $\theta \in \Theta$, $x \in X$ and $t \geq 0$.

For $t \to \infty$ we get that

$$\int_0^\infty \|\Phi(\theta,\tau)x\|^2 d\tau \leq K\|x\|^2, \qquad \text{for all } \theta \in \Theta \text{ and } x \in X.$$

From [15], Lemma 2.4. it results that the linear skew-product semiflow $\pi = (\Phi, \sigma)$ is exponentially stable.

□

Necessary condition

In what follows it will be presented the necessary condition, which needs a stronger hypothesis.

Theorem 2.2. Let $\pi = (\Phi, \sigma)$ be a linear skew-product semiflow (LSPS) exponentially stable. Then for all $\Gamma \in \mathcal{B}^+(X, X')$ with the property that there exists $\gamma > 0$ such that $\Gamma x(x) \geq \gamma \|x\|^2$, for all $x \in X$, there exists $W : \Theta \to \mathcal{B}^+(X, X')$ bounded such that

$$W(\sigma(\theta, t))\Phi(\theta, t)x(\Phi(\theta, t)x) + \int_0^t \Gamma(\Phi(\theta, \tau)x)(\Phi(\theta, \tau)x)d\tau = W(\theta)x(x), \qquad (2.2)$$

for all $t \geq 0$, $\theta \in \Theta$ and $x \in X$.

Proof. We consider $x, y \in X$, $\theta \in \Theta$ and

$$W(\theta)x(y) = \int_0^\infty \Gamma(\Phi(\theta,\tau)x)(\Phi(\theta,\tau)y)d\tau. \qquad (2.3)$$

It can be shown by computations that $W \in \mathcal{B}^+(X, X')$.

We have now that

$$W(\sigma(\theta,t))\Phi(\theta,t)x(\Phi(\theta,t)x) = \int_0^\infty \Gamma(\Phi(\sigma(\theta,t),\tau)\Phi(\theta,t)x)(\Phi(\sigma(\theta,t),\tau)\Phi(\theta,t)x)d\tau =$$

$$= \int_0^\infty \Gamma(\Phi(\theta,t+\tau)x)(\Phi(\theta,t+\tau)x)d\tau = \int_0^\infty \Gamma(\Phi(\theta,\tau)x)(\Phi(\theta,\tau)x)d\tau - \int_0^t \Gamma(\Phi(\theta,\tau)x)(\Phi(\theta,\tau)x)d\tau =$$

$$= W(\theta)x(x) - \int_0^t \Gamma(\Phi(\theta,\tau)x)(\Phi(\theta,\tau)x)d\tau$$

and therefore we get the relation (2.2) and the proof is complete. □

As a result of the last two theorems, we now obtain the necessary and sufficient conditions for the exponential stability of a linear skew-product semiflow (LSPS), as follows:

Corollary 2.1. The linear skew-product semiflow $\pi = (\Phi, \sigma)$ is exponentially stable if and only if for all $\Gamma \in \mathcal{B}^+(X, X')$ with the property that there exists $\gamma > 0$ such that $\Gamma x(x) \geq \gamma \|x\|^2$, for all $x \in X$, there exists $W : \Theta \to \mathcal{B}^+(X, X')$ bounded such that

$$W(\sigma(\theta,t))\Phi(\theta,t)x(\Phi(\theta,t)x) + \int_0^t \Gamma(\Phi(\theta,\tau)x)(\Phi(\theta,\tau)x)d\tau = W(\theta)x(x), \qquad (2.4)$$

for all $t \geq 0$, $\theta \in \Theta$ and $x \in X$.
Proof. *Necessity* results from Theorem 2.2.

Sufficiency results analogously with Theorem 2.1., by considering in addition $\Gamma \in \mathcal{B}^+(X, X')$ with the same property as in Theorem 2.2.

Discrete version

In what follows we will also present the discrete versions of the above results.

A sufficient condition is given as follows:

Theorem 2.3. Let $\pi = (\Phi, \sigma)$ be linear skew-product semiflow. If there exists $W : \Theta \to \mathcal{B}^+(X, X')$ bounded such that

$$W(\sigma(\theta, n))\Phi(\theta, n)x(\Phi(\theta, n)x) + \sum_{k=0}^{n-1} \|\Phi(\theta, k)x\|^2 \leq W(\theta)x(x), \qquad (2.5)$$

for all $\theta \in \Theta$, $n \in \mathbb{N}^*$ and $x \in X$, then the linear skew-product semiflow is exponentially stable.

The sufficient condition is given in the following theorem:

Theorem 2.4. Let $\pi = (\Phi, \sigma)$ be a linear skew-product semiflow (LSPS) exponentially stable. Then for all $\Gamma \in \mathcal{B}^+(X, X')$ with the property that there exists $\gamma > 0$ such that $\Gamma x(x) \geq \gamma \|x\|^2$, for all $x \in X$, there exists $W : \Theta \to \mathcal{B}^+(X, X')$ bounded such that

$$W(\sigma(\theta, n))\Phi(\theta, n)x(\Phi(\theta, n)x) + \sum_{k=0}^{n-1} \Gamma(\Phi(\theta, k)x)(\Phi(\theta, k)x) = W(\theta)x(x), \qquad (2.6)$$

for all $n \in \mathbb{N}^*$, $\theta \in \Theta$ and $x \in X$.

As a result of Theorems 2.3. and 2.4. it can be obtained the following corollary:

Corollary 2.2. The linear skew-product semiflow $\pi = (\Phi, \sigma)$ is exponentially stable if and only if for all $\Gamma \in \mathcal{B}^+(X, X')$ with the property that there exists $\gamma > 0$ such that $\Gamma x(x) \geq \gamma \|x\|^2$, for all $x \in X$, there exists $W : \Theta \to \mathcal{B}^+(X, X')$ bounded such that

$$W(\sigma(\theta, n))\Phi(\theta, n)x(\Phi(\theta, n)x) + \sum_{k=0}^{n-1} \Gamma(\Phi(\theta, k)x)(\Phi(\theta, k)x) = W(\theta)x(x), \tag{2.7}$$

for all $n \in \mathbb{N}^*$, $\theta \in \Theta$ and $x \in X$.
Proof. *Necessity* results from Theorem 2.4.

Sufficiency results analogously with Theorem 2.3. □

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Thank you for your attention!

International Workshop
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on Functional Analysis
Timişoara, October 12-14, 2012