An extension of a Lyapunov operator equation to the case of variational equations

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Outline

We present a characterization for the uniform exponential stability of linear skew-product three-parameter semiflows, generated by a new class of evolution equations. We extend, thus, a classical result of R. Datko from the 70s:

R. Datko, 1973

An evolution family $\{U(t,t_0)\}_{t\geq t_0\geq 0}$ is uniformly exponentially stable (i.e. there exist $N,\nu>0$ such that $||U(t,t_0)||\leq Ne^{-\nu(t-t_0)},\ \forall\ t\geq t_0\geq 0$) if and only if there exist $k,p>0$ such that:

$$\left(\int_t^{\infty}||U(\tau,t)x||^p d\tau\right)^{\frac{1}{p}}\leq k||x||,\ \text{for all } t\geq 0 \text{ and } x\in X,$$

where X is a Banach space.

M. Megan, C. Stoica (2007)

M. Rasmussen (2009)

P. Viet Hai (2010)

Three-parameter semiflows

Let us consider (Θ, d) a metric space, X a Banach space, $\mathcal{B}(X)$ the space of all bounded operators acting on X and $\Delta = \{(t, t_0) \in \mathbb{R}^2 : t \geq t_0 \geq 0\}$. We denote the norm of vectors on X and operators on $\mathcal{B}(X)$ by $||\cdot||$.

Definition 1.1

A three-parameter semiflow $\sigma : \Theta \times \Delta \to \Theta$ *is defined by the properties:*

i) $\sigma(\theta, t, t) = \theta$ *for all* $t \geq 0$ *and all* $\theta \in \Theta$

ii) $\sigma(\sigma(\theta, s, t_0), t, s) = \sigma(\theta, t, t_0)$ *for all* $t \geq s \geq t_0 \geq 0$ *and all* $\theta \in \Theta$.

If in addition $(\theta, t, t_0) \mapsto \sigma(\theta, t, t_0)$ *is continuous, then* σ *is called a continuous three-parameter semiflow on* Θ.

Linear skew-product three-parameter semiflows

Definition 1.2

The pair $\pi = (\Phi, \sigma)$ *is said to be a linear skew-product three-parameter semiflow on* X *if* $\Phi : \Theta \times \Delta \to \mathcal{B}(X)$ *satisfies the following properties:*

i) $\Phi(\theta, t, t) = I$ *for all* $t \geq 0$ *and all* $\theta \in \Theta$*, where* I *represents the identity operator on* X*;*

ii) $\Phi(\sigma(\theta, s, t_0), t, s)\Phi(\theta, s, t_0) = \Phi(\theta, t, t_0)$*, for all* $t \geq s \geq t_0 \geq 0$ *and all* $\theta \in \Theta$*.*

iii) the maps $(\theta, t, t_0) \mapsto \Phi(\theta, t, t_0)x$ *are continuous on* $\Theta \times \Delta$ *for each* $x \in X$*;*

iv) there exist $M, \omega \in \mathbb{R},\ M \geq 1$ *such that*

$$||\Phi(\theta, t, t_0)|| \leq Me^{\omega(t-t_0)},\ \textit{for all}\ t \geq t_0 \geq 0,\ \textit{and}\ \theta \in \Theta.$$

The classical example of a linear skew-product three-parameter semiflow arises as the solution operator for a variational equation.

Example 1.1

Let σ be a continuous three-parameter semiflow on the metric space Θ and $A : \Theta \to \mathcal{B}(X)$ a continuous map. If $\Phi(\theta, \cdot, \cdot)$ solves the variational equation

$$\dot{x}(t) = A(\sigma(\theta, t, t_0))x(t), \ \ t \geq t_0,$$

then the pair $\pi = (\Phi, \sigma)$ is a linear skew-product three-parameter semiflow.

It is easy to prove that linear skew-product three-parameter semiflows are generalizations of the classical concept of evolution families.

Example 1.2

If $\{U(t,t_0)\}_{t \geq t_0 \geq 0}$ *is an evolution family on* X *and* σ *is a three-parameter semiflow on* Θ*, then* $\pi = (\Phi, \sigma)$ *is a linear skew-product three-parameter semiflow, where*

$$\Phi(\theta, t, t_0) = U(t, t_0).$$

Thus, we can consider that evolution families are particular cases of linear skew-product three-parameter semiflows. Conversely, considering $\Theta = \mathbb{R}_+$*,*
$\sigma : \mathbb{R}_+ \times \Delta \to \mathbb{R}_+,\ \sigma(\theta, t, t_0) = \theta$*, and* $\pi = (\Phi, \sigma)$ *a linear skew-product three-parameter semiflow on* X*, we have that* $\{U(t,t_0)\}_{t \geq t_0 \geq 0},\ U(t, t_0) = \Phi(0, t, t_0)$ *is an evolution family on* X*.*

Definition 2.1

Let $\pi = (\Phi, \sigma)$ *be a linear skew-product three-parameter semiflow.* $\pi = (\Phi, \sigma)$ *is said to be uniformly exponentially stable if there exist* $N \geq 1$ *and* $\nu > 0$ *such that:*

$$||\Phi(\theta, t, t_0)x|| \leq Ne^{-\nu(t-t_0)}||x||, \ \forall\, x \in X, \ \forall\, t \geq t_0 \geq 0, \ \forall\, \theta \in \Theta.$$

We consider the set $\mathcal{L}(X,X') = \{A : X \to X' : \ Ax(x) \geq 0, \ \forall\, x \in X\}$, where X is a Banach space, and X' is its topological dual.

Theorem 2.1

Let $\pi = (\Phi,\sigma)$ be a linear skew-product three-parameter semiflow. $\pi = (\Phi,\sigma)$ is uniformly exponentially stable if and only if for all $\Gamma \in \mathcal{L}(X,X')$ with the property that there exists $\gamma > 0$ such that $\Gamma x(x) \geq \gamma ||x||^2$, for all $x \in X$, there exists $W : \Theta \times \mathbb{R}_+ \to \mathcal{L}(X,X')$ bounded, such that:

$$
\begin{aligned}
&(W(\sigma(\theta,t,t_0),t)\Phi(\theta,t,t_0)x)(\Phi(\theta,t,t_0)x)+\\
&\quad + \int_{t_0}^{t} \Gamma(\Phi(\theta,\tau,t_0)x)(\Phi(\theta,\tau,t_0)x)d\tau = (W(\theta,t_0)x)(x), \quad (1)
\end{aligned}
$$

for all $t \geq \tau \geq t_0 \geq 0$, $(\theta,x) \in \Theta \times X$.

sketch of proof

For the *necessity* part we consider

$$(W(\sigma(\theta,t,t_0),t)x)(y) = \\ = \int_t^{\infty} \Gamma(\Phi(\sigma(\theta,t,t_0),\tau,t)x)(\Phi(\sigma(\theta,t,t_0),\tau,t)y)d\tau.$$

and we show that relation (1) holds.
For the *sufficiency* part we use some Datko-type auxiliary result extended to the case of linear skew-product three-parameter semiflows, namely the following

Theorem 2.2

M. Megan, C. Stoica, 2009

$\pi = (\Phi, \sigma)$ is uniformly exponentially stable if and only if there exist $p > 1$ and $\tilde{N} \geq 1$ such that:

$$\int_{t_0}^{\infty} ||\Phi(s, t_0, x)v||^p ds \leq \tilde{N}^p ||v||^p, \ \forall\, (t, t_0, x, v) \in T \times Y,$$

where $T = \{(t, t_0) \in \mathbb{R}^2_+ : \ t \geq t_0\}$, and $Y = X \times V$, with (X, d) a metric space and V a Banach space.

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3. M. Rasmussen. *Dichotomy spectra and Morse decompositions of linear nonautonomous differential equations*, J. Diff. Eqns. **246** (2009), 2242-2263.

4. C. Stoica, M. Megan *On uniform exponential stability for skew-evolution semiflows on Banach spaces*, Nonlinear Analysis, **7**2 (2010), 1305-1313.