Spectral order and its morphisms

Artur Płaneta

University of Agriculture in Krakow, Poland

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- $\boldsymbol{B}_s(\mathcal{H}) = \{A \in \boldsymbol{B}(\mathcal{H})\colon A = A^*\}$

The definition of spectral order

- Let A and B be selfadjoint operators in $\mathcal{H}$ with spectral measure E_A and E_B, respectively, we write $A \preccurlyeq B$ if $E_B((-\infty, x]) \leqslant E_A((-\infty, x])$ for all $x \in \mathbb{R}$.

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- This definition was introduced in 1971 by Olson.

Lattices

- Kadison (1951): $(\boldsymbol{B}_s(\mathcal{H}), \leqslant)$ is an anti-lattice, i.e., for any $A, B \in \boldsymbol{B}_s(\mathcal{H})$, the supremum of the set $\{A, B\}$ exists if and only if A, B are comparable (either $A \leqslant B$ or $B \leqslant A$).

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- Sherman (1951): If the set of all selfadjoint elements of a C^*-algebra $\mathcal{A}$ with the usual order forms a lattice, then $\mathcal{A}$ is commutative.
- Olson (1971): If $\mathcal{S}$ is the set of all selfadjoint elements of a von Neumann algebra $\mathscr{V}$ in $\boldsymbol{B}(\mathcal{H})$ then, $(\mathcal{S}, \preccurlyeq)$ is a conditionally complete lattice.

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- $E_{\mathbf{A}}$-joint spectral measure of $\mathbf{A} = (A_1, \ldots, A_\kappa)$,

Definition

Let $\mathbf{A} = (A_1, \ldots, A_\kappa)$ and $\mathbf{B} = (B_1, \ldots, B_\kappa)$ be a κ-tuples of commuting selfadjoint operators in $\mathcal{H}$. We write $\mathbf{A} \preccurlyeq \mathbf{B}$ if $E_{\mathbf{B}}((-\infty, x]) \leqslant E_{\mathbf{A}}((-\infty, x])$ for every $x = (x_1, \ldots, x_\kappa) \in \mathbb{R}^\kappa$, where $(-\infty, x] := (-\infty, x_1] \times \ldots \times (-\infty, x_\kappa]$.

Definitions

- Let $\iota \in \{1, \ldots, \kappa\}$. We define a relation $\leqslant_\iota$ on $\mathbb{R}^\kappa$ requiring that $a \leqslant_\iota b$ if $a_j \leqslant b_j$ for $j = 1, \ldots, \iota$ and $a_j = b_j$ for $j = \iota + 1, \ldots, \kappa$. If $\iota = \kappa$ we write $\leqslant$ instead of $\leqslant_\kappa$.

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- Let $\kappa \in \mathbb{N}_*$, $\Omega \subset \mathbb{R}^\kappa$ and $\varphi \colon \Omega \to \overline{\mathbb{R}}$. We say, that φ is a separately increasing function if

$$x \leqslant y \Rightarrow \varphi(x) \leqslant \varphi(y),$$

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$$x \leqslant y \Rightarrow \varphi(x) \leqslant \varphi(y),$$

  for every $x, y \in \Omega$.
- Let $(X, \leq)$ be a partially ordered set. A set $S \subset X$ is called a lower set in X if

$$(y \leq x \wedge x \in S) \Rightarrow y \in S,$$

  for every $x, y \in X$.

Lemma

Let $\boldsymbol{A} = (A_1, \ldots, A_\kappa)$ *and* $\boldsymbol{B} = (B_1, \ldots, B_\kappa)$ *be a commuting* κ*-tuple of selfadjoint operators in* $\mathcal{H}$ *and let* $\iota \in \{1, \ldots, \kappa\}$*. Assume that* $A_j = B_j$ *for every* $j = \iota + 1, \ldots, \kappa$*. If* $\boldsymbol{A} \preccurlyeq \boldsymbol{B}$*, then*

$$E_{\boldsymbol{B}}(\Omega) \leqslant E_{\boldsymbol{A}}(\Omega) \tag{1}$$

for every $\Omega \in \mathfrak{B}(\mathbb{R}^\kappa)$*, which is a lower set in* $(\mathbb{R}^\kappa, \leqslant_\iota)$*.*

Theorem

Let $\boldsymbol{A} = (A_1, \ldots, A_\kappa)$ *and* $\boldsymbol{B} = (B_1, \ldots, B_\kappa)$ *be commuting* κ*-tuple of selfadjoint operators in* $\mathcal{H}$*. Then the following conditions are equivalent:*

(i) $\mathbf{A} \preccurlyeq \mathbf{B}$,

(ii) $\varphi(\mathbf{A}) \preccurlyeq \varphi(\mathbf{B})$ *for every separately increasing function* $\varphi \in S(\mathbb{R}^\kappa, E_{\boldsymbol{A}}) \cap S(\mathbb{R}^\kappa, E_{\boldsymbol{B}})$,

(iii) $A_j \preccurlyeq B_j$ *for every* $j = 1, \ldots, \kappa$.

Remark

Suppose that $\dim \mathcal{H} \geqslant 1$. Then each Borel function $\varphi\colon \mathbb{R}^{\kappa} \to \mathbb{R}$ satisfying condition

$$\mathbf{A} \preccurlyeq \mathbf{B} \implies \varphi(\mathbf{A}) \preccurlyeq \varphi(\mathbf{B}) \tag{2}$$

for every **A**, **B** κ-tuples of commuting selfadjoint operators, has to be separately monotonically increasing.

Corollary

Let $\mathbf{A}$ *and* $\mathbf{B}$ *be* κ*-tuples of commuting selfadjoint operators. Then the following conditions are equivalent:*

(i) $\mathbf{A} \preccurlyeq \mathbf{B}$,

(ii) $\varphi(\mathbf{A}) \leqslant \varphi(\mathbf{B})$ *for every separately monotonically increasing bounded continuous function* $\varphi\colon \mathbb{R}^{\kappa} \to \mathbb{R}$,

(iii) $\varphi(\mathbf{A}) \leqslant \varphi(\mathbf{B})$ *for every separately monotonically increasing bounded Borel function* $\varphi\colon \mathbb{R}^{\kappa} \to \mathbb{R}$.

Effects and automorphisms

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- $\mathcal{E}(\mathcal{H}) = \{A \in \boldsymbol{B}_s(\mathcal{H})\colon 0 \leqslant A \leqslant I.\}$
- $\mathcal{E}^\kappa(\mathcal{H}) = \{\mathbf{A} \in \mathcal{E}(\mathcal{H})^\kappa\colon \mathbf{A}$ is commutative $\}$,
- $\mathrm{Aut}(\mathcal{E}(\mathcal{H}), \preccurlyeq)$ (respectively $\mathrm{Aut}(\boldsymbol{B}_s(\mathcal{H}), \preccurlyeq)$) is a set of bijective selfmaps of $\mathcal{E}(\mathcal{H})$ (resp. $\boldsymbol{B}_s(\mathcal{H})$) which satisfy $A \preccurlyeq B \Leftrightarrow \phi(A) \preccurlyeq \phi(B)$ for every $A, B \in \mathcal{E}(\mathcal{H})$ (resp. $\boldsymbol{B}_s(\mathcal{H})$).

Examples of automorphisms of $\mathcal{E}^\kappa(\mathcal{H})$

- Mapping $\mathbf{A} \to U\mathbf{A}U^* = (UA_1U^*, \ldots, UA_\kappa U^*)$, where U is an unitary or antiunitary operator on $\mathcal{H}$.

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- Mapping $\mathbf{A} \to \psi_T(\mathbf{A}) = (\int_{\mathbb{R}} \lambda_j dE^T_{\mathbf{A}})^\kappa_{j=1}$, where $T\colon \mathcal{H} \to \mathcal{H}$ is a bijective bounded linear or conjugate operator on $\mathcal{H}$, if $\dim \mathcal{H} = \infty$ or a bijective semilinear operator, if $\dim \mathcal{H} < \infty$. The spectral measure $E^T_{\mathbf{A}}$ is given by the following resolution of the identity $x \to I - P_{T(\mathscr{R}(E_{\mathbf{A}}(x,\infty)))}$, $x \in \mathbb{R}^\kappa$.

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Theorem

For every automorphism ϕ of $(\mathcal{E}^\kappa(\mathcal{H}), \preccurlyeq)$, there exists an automorphism ϕ_1 and bijective separately increasing Borel function $f : [0,1]^\kappa \to [0,1]^\kappa$ such that

$$\phi(A) = \phi_1(f(A)), \quad A \in \mathcal{E}^\kappa(\mathcal{H}),$$

and ϕ_1 has homogeneity property i.e. $\phi_1(\lambda \boldsymbol{P}) = \lambda \phi_1(\boldsymbol{P})$ for $\lambda \in [0,1]$ and commuting κ-tuple of orthogonal projection $\boldsymbol{P}$.

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