$\mathcal{H}^2$-inner functions

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Following Hedenmalm, Lindqvist and Seip (Duke Math. J. 86 (1997) 1-37) we consider the the space $\mathcal{H}^2$ of all Dirichlet series of the form

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \tag{1}$$

with finite norm

$$\|f\|_{\mathcal{H}^2}^2 = \sum_{n=1}^{\infty} |a_n|^2 < +\infty.$$

The space $\mathcal{H}^2$ is a Hilbert space of analytic functions in the half-plane

$$\mathbb{C}_{1/2} = \{s \in \mathbb{C} : \ \Re s > 1/2\}$$

(Cauchy-Schwarz inequality).

A characteristic feature of the space $\mathcal{H}^2$ is that of its reproducing kernel function being essentially the Riemann zeta function from number theory:

$$K_{\mathcal{H}^2}(s, s') = \zeta(s + \bar{s'}), \quad s, s' \in \mathbb{C}_{1/2},$$

where $\zeta(s) = \sum_{n \geq 1} 1/n^s$.

For every positive integer $n \in \mathbb{Z}^+$ we have a natural operator $S(n)$ acting on $\mathcal{H}^2$ defined by

$$S(n)f(s) = n^{-s} f(s), \quad s \in \mathbb{C}_{1/2}, \qquad (2)$$

for functions $f \in \mathcal{H}^2$.

This provides us with a function

$$S : \mathbb{Z}^+ \ni n \mapsto S(n)$$

which is easily seen to be a multiplicative semigroup of operators in the sense that

$$S(nm) = S(n)S(m), \quad n, m \in \mathbb{Z}^+,$$

and $S(1) = I$, where I is the identity operator.

We call this semigroup S the shift semigroup on $\mathcal{H}^2$.

One can think of the shift semigroup as a counterpart for the space $\mathcal{H}^2$ of the classical shift operator on the Hardy space $H^2(\mathbb{D})$ given by multiplication by the complex coordinate.

By an $\mathcal{H}^2$-inner function we mean a function φ in $\mathcal{H}^2$ of unit norm such that

$$S(n)\varphi \perp S(m)\varphi \quad \text{in } \mathcal{H}^2$$

for all integers $n, m \in \mathbb{Z}^+$ with $n \neq m$.

In a recent paper (Acta Math. Hungar. 128 (2010) 265-286) we have characterized a certain natural class of shift invariant subspaces $\mathcal{I}$ of $\mathcal{H}^2$ of the form

$$\mathcal{I} = \varphi \mathcal{H}^2,$$

where φ is $\mathcal{H}^2$-inner (double commutativity of restricted shift operators).

Problem 1. *Describe the $\mathcal{H}^2$-inner functions.*

In view of recent work on Bergman spaces Problem 1 might amount to calculate the associated characteristic operator functions or transfer functions for associated systems.

Adapting such an approach the following might be considered:

Problem 2. *Describe the parts of the adjoint shift semigroup*

$$S^* : \mathbb{Z}^+ \ni n \mapsto S(n)^* \in \mathcal{L}(\mathcal{H}^2).$$

Recall the classical result that the parts of the adjoint shift operator on a vector-valued Hardy space $H^2(\mathbb{D}, \mathcal{E})$ are the $C_0.$ contractions (Sz.-Nagy, Foias and others).