FIXED POINT TECHNIQUE FOR A CLASS OF BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

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Outline

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Stochastic Differential Equations - SDE

- $$dX = f(t, X(t))dt + g(t, X(t))dW_t$$
  where $\{W_t\}$ a d-dimensional Brownian motion.
- A solution is an adapted stochastic process $X : [a, b] \times \Omega \to R$ which check the above relation at any time moment $t \in [t_0, T]$, verify a given initial condition $X(t_0) = X_0$ (X_0 is a random variable !) and is $\mathscr{F}_t$-measurable on a filtered probability space $(\Omega, \mathscr{F}, P, \{\mathscr{F}\})$.

Example

the price of a stock

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad S_0 > 0,\ \mu \in R,\ \sigma > 0$$

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Backward Stochastic Differential Equation - BSDE

- $$dY = f(t, Y(t), Z(t))dt - Z(t)dW_t, \;\; Y_1 = \xi$$

- In the field of control, we usually regard $Z(\cdot)$ as an adapted control and $Y(\cdot)$ as the state of the system. We are allowed to choose an adapted control $Z(\cdot)$ which drives the state $Y(\cdot)$ of the system to the given target ξ at time $t = 1$. This is so-called reachability problem. So in fact we are looking for a pair of stochastic processes $\{Y(t), Z(t), 0 \leq t \leq 1\}$ with values in $R \times R$ which is $\mathscr{F}_t$-adapted and satisfies the above equation. Such pair is called an adapted solution of the equation.

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Applications of the BSDE's

- Control and filtering theory
- Solving of non-linear PDE (using the famous Feynman-Kac formula)
- Finances - pricing of the derivative assets (the famous Black -Scholes formula)

Example

the price of an European call option

$$dC_t = [(\mu - r)S_t g(S_t, t) + rC_t]dt + \sigma S_t g(S_t, t)dW_t,$$

with $C_T = \max(S_t - K, 0)$ and $(W_t)_{t\in[0,T]}$ is a one-dimensional standard Brownian motion.

Introduction

A short history

- Pardoux and Peng [25] in (1990) showed the existence and uniqueness of the adapted solution under the condition that $f(t,y)$ is uniformly Lipschitz continuous in (y).
- Since then, the interest of the BSDEs, has increased steadily (see Antonelli, 1993, Peng, 1993 or Ma, Protter and Yong 1994), due to the connections of this subject with mathematical finance, stochastic control, and partial differential equations.
- In particular, many efforts have been made to relax the assumptions on the coefficient functions (for instance Mao, 1995, Lepltier and San Martin, 1998, Pardoux and Rascnau 1998, Lepeltir and San Martin 2002, Kobylanski 2000, Briand and Hu 2006, Buckdahn, Engelbert and Rascanu or Negrea and Preda 2011).
- An interesting domain for the applications of this class of stochastic differential equations, mainly in mathematical finance, have also been found (see for example, Karatzas and Shreve 1991, Cvitanic and Karatzas 1993, El Karoui , Peng and Quenez 1997 or Hu, Imkeller and

Introduction

Weaker conditions

- The starting point of our equation is given by S. Athanassov [2], where is proved a uniqueness theorem of Nagumo type for the Cauchy problem generalizing several known uniqueness theorems and sufficient conditions to guarantee the convergence of the Picard successive approximations for the ordinary differential equations.
- The importance of the Athanassov's result comes from the fact that the coefficients functions can have some singularities at the time moment $t = 1$. That results clarifies the relationship between the modulus of the continuity and growth conditions imposed to the coefficient functions.
- Stochastic generalizations of the results of Athanassov for SDE's are given by A. Constantin (see [5], [6], [7]) or Gh. Constantin [8], Gh. Constantin and Negrea[9], or Negrea and Preda [23] and others.

Introduction

Weaker conditions

- A fixed point method to show the existence of the solution for a BSDE was proposed still to the first papers on this frame, as for example the papaer of Antonelli [1], more other mathematicians try to apply different fixed point theorem for some classes of BSDEs. The Schauder's fixed theorem which is an interesting extension of the Brouer's fixed theorem in the case of infinite dimensional spaces. From our best knowledge the first intention to apply this theorem on the frame of BSDE was proposed by Negrea [22]. Others results on this idea were given by Negrea and Preda [23] and [24].

Preliminaries

Frame of BSDE

- Let $\{B_t\}_{0\leq t\leq 1}$ denote an d-dimensional Brownian motion defined on some complete probability space $(\Omega, \mathscr{F}, P)$ with the natural filtration $\{\mathscr{F}_t, 0 \leq t \leq 1\}$ and $\xi(\omega)$ is a given $\mathscr{F}_1$-measurable random variable with $E(|\xi|^2) < \infty$.
- If $\mathscr{P}$ is the σ-algebra of $\mathscr{F}_t$-progressively measurable subsets of $[0,1]\times\Omega$ and M^2 is the family of real-valued processes which are $\mathscr{P}$-measurable and square integrable with respect $\lambda \times P$, λ being the Lebesgue measure.

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Preliminaries

Frame of BSDE

- A solution of a backward stochastic differential equation is a pair of stochastic processes $\{Y(t,\omega), Z(t,\omega) : \ t \in [0,1]\} \in \times M^2([0,1], R^m) \times M^2([0,1], R^{m\times d})$.
- For $Y \in M^2([0,1], R^m)$ we define the norm $|Y|^2 = \sum_{j=1}^{m} E[\sup_{0\leq t\leq 1} |Y_j(t)|^2]$.
  Also, for $Z \in M^2([0,1], R^{m\times d})$ we define the norm $|Z|^2 = \sum_{j=1}^{m} E[\int_0^1 |Z_j(t)|^2 dt]$.

Gronwall's type Lemma

- Next, in a some similar way as Athanassov [2], we given in [23] the following lemma:

Lemma 2.1.

Let $u(t)$ *be a continuous, positive function on* $0 < t < 1$ *having nonnegative derivative* $u'(t) \in L([0,1])$*. Let* $v(t)$ *be a continuous, nonnegative function for* $0 \le t \le 1$ *such that* $v(t) = o(u(t))$ *as* $t \to 1^-$ *and*

$$v(t) \le \int_t^1 \frac{u'(s)}{u(s)} v(s) ds, \ \forall\, 0 \le t \le 1.$$

Then $v(t) \equiv 0$ *on* $0 \le t \le 1$*.*

Assumptions

- We consider the following general backward stochastic differential equation

BSDE

$$Y(t) + \int_t^1 f(s, Y(s), Z(s))ds - \int_t^1 Z(s)dW(s) = \xi, \qquad (1)$$

$0 \le t \le 1$, with $f : \Omega \times (0,1) \times R^m \times R^{m \times d}$

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Assumptions

Hypothesis

i) f is a $\mathscr{P} \otimes \mathscr{B}_{\mathbb{R}^m} \otimes \mathscr{B}_{\mathbb{R}^{m\times d}}$ measurable and $\mathscr{F}_t$-adapted function and is continuous function in the variable (y,z) on $M^2([0,1],\mathbb{R}^m) \times M^2([0,1],\mathbb{R}^{m\times d})$;

Hypothesis

ii) $f(\cdot,0,0)$ is in $M^2([0,1],\mathbb{R}^m) \times M^2([0,1],\mathbb{R}^{m\times d})$;

Assumptions

Hypothesis

iii) there exists a continuous, positive and derivable function $u(t)$ on $0 \le t \le 1$, having nonnegative derivative $u'(t) \in L([0,1])$, such that

$$|f(t,y_1,z_1) - f(t,y_2,z_2)|^2 \le \frac{u'(t)}{A_1 u(t)}(|y_1 - y_2|^2 + \|z_1 - z_2\|^2), \qquad (2)$$

for all $y_1, y_2 \in \mathbb{R}^m, z_1, z_2 \in \mathbb{R}^{m\times d}$ $0 \le t \le 1$, and A_1 is a positive real constant;

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Assumptions

Hypothesis

iv) with the same function $u(t)$ as above,

$$|f(t,y,z)|^2 \leq \frac{u'(t)}{A_2}(1+|y|^2+\|z\|^2), \quad (3)$$

for all $y \in R^m, z \in R^{m\times d}$, $0 \leq t \leq 1$, and A_2 is a positive real constant;

Hypothesis

v) ξ is a given $\mathscr{F}_1$-measurable random variable such that $E|\xi|^2 < \infty$ and $\alpha \in R^n$.

Assumptions

- It's not difficult to see that any Lipschitz function f verifies the above assumption. Also, any function with a quadratic growth satisfies the assumption **iv)**. Moreover, the assumption **iv)** not required the continuity of the coefficient functions at the final moment $t = 1$ because we suppose just $u'(t) \in L([0,1])$, hence we will prove a similar Nagumo's type theorem for BSDE's frame. We will see, in the last section, an example where the coefficient function f can have a jump in a neighborhood of $t = 1$, and this situation is frequent observed in applications of BSDE's to the financial modeling (see for example [13], [14], [11], [10].
- An extension of this results for a class of forward-backward stochastic differential equations is given in Negrea and Preda, 2012 [24].

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Existence and uniqueness

Theorem 3.1.

Let f satisfies the above hypotheses and $\xi \in L^2(\Omega, \mathscr{F}_1, P, R)$, then there exists a unique pair $(Y,Z) \in M^2([0,1], \mathbb{R}^m) \times M^2([0,1], \mathbb{R}^{m \times d})$ which satisfies the equation (1) on any compact subsets of interval $[0,1]$.

Main results

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Sketch of proof

Uniqueness. Let (Y_1, Z_1) and (Y_2, Z_2) be two solutions in $M^2((0,1), R^m) \times M^2([0,1], R^{m\times d})$ of the equation (1.1).

- $$Y_1(t) - Y_2(t) + \int_t^1 Z_1(s)dB_s - \int_0^t Z_2(s)dB_s =$$
  $$= \int_t^1 [f(Y_1(s), Z_1(s)) - f(Y_2(s), Z_2(s))] \, .ds$$
- For any $0 < t < 1$, we denote
  $$v(t) = \sup_{s\geq t}(|Y_1(s) - Y_2(s)|^2 + \|Z_1(s) - Z_2(s)\|^2), \quad 0 < t_0 \leq t \leq s < 1$$
- and from the hypothesis (iii), Lemma 2.1 we obtain
  $$v(t) \equiv 0, \quad 0 < t < 1.$$

Main results

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Sketch of proof

Existence.

- Let to prove the existence of a solution of system (1) on some interval $[\delta_1, \delta_2]$ with positive numbers $0 < \delta_1 < \delta_2 < 1$ which will be explained from along to the proof. We will made a reasoning similar as in [9].
- Let the Banach space $B^2 = ((M^2([0,1], R^m) \times M^2([0,1], R^{m\times d})$ with the norm

$$|||(y,z)||| = \sqrt{|y|^2 + \|z\|^2},\ |x|^2 = E[\sup_{0\le t\le 1} |y(t)|^2],\ \|z\|^2 = E[\int_t^1 |z|^2].$$

- Let $q = |\xi|$ and $Q = 2q$. We define the set

$$\mathscr{S} = \{(Y,Z) \in B^2 :\ |Y| \le Q,\ \|Z\| \le Q, \text{ for } t \in [\delta_1, \delta_2]\}$$

  which it is a closed bounded and convex subset of the Banach space $B^2(0,1), |||.|||$.

Main results

Sketch of proof

Existence.

- We will define a map $T : B^2 \to B^2$ such that $(Y,Z) \in B^2$ is a solution of the BSDE (1) if it is a fixed point of T.
- Let $(U,V) \in \mathscr{S}$, and $(Y,Z) = T(U,V)$ with $\{Y_t\}$ given by the relation (1) and $\{Z_t\}$ is obtained by using Ito's martingale representation theorem, applied to the square integrable random variable

$$\xi + \int_t^1 f(s,U_s,V_s)ds, \;\; t \in [\delta_1, \delta_2]$$

  This relation will prove that $(Y,Z) \in B^2$ is solution of the BSDE if it will a fixed point of T

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Main results

Sketch of proof

Existence.

- First, we use the hypothesis (iv) and will prove that

$$T(\mathscr{S}) \subseteq \mathscr{S} \ .$$

- Next, we will prove that the set $T(\mathscr{S})$ is equicontinuous.
- For (U,V) and $(U',V') \in \mathscr{S}$ we evaluate

$$|||T(U(t),V(t)) - T(U'(t),V'(t))|||^2 = |Y(t) - Y'(t)|^2 + \|Z(t) - Z'(t)\|^2, \ t \in [\delta_1, \delta_2] \ .$$

  and then from the hypothesis (iii), the continuity of the function f in the variables y,z on B^2 we deduce by Lebesgue convergence theorem that T is continuous.
- Applying the Schauder's fixed point theorem we obtain that T has a fixed point $\mathscr{S}$, thus the stochastic differential system (1) has a solution on $[\delta_1, \delta_2]$, for any positive real numbers

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Main results

Regularity properties

Now, we will give some results on the stability properties of the solution of the equation (1). We consider the families of backward stochastic integral equation

$$Y_\lambda(t) = \xi_\lambda + \int_t^1 f_\lambda(s, Y_\lambda(s), Z_\lambda(s))ds - \int_t^1 Z_\lambda(s)dB_s, \quad 0 \le t \le 1, \tag{4}$$

whit $\lambda \in \Lambda$- a open and bounded set $\subset R^n$.
First, under the considered hypothesis, we prove the existence and uniqueness of solutions and the continuity with respect to the final condition ξ in the equation (4).

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Main results

Theorem 3.2.

If, for any $\lambda \in \Lambda$, the coefficient functions f_λ satisfy the hypothesis (i)-(v), then the family (4) has a unique solution $(Y_\lambda, Z_\lambda) \in M^2([0,1]), M^2([0,1], \mathbb{R}^m) \times M^2([0,1], \mathbb{R}^{m\times d})$.
Moreover, if

$$\lim_{k\to\infty} |\xi_{\lambda,k} - \xi_\lambda|^2 = 0,$$

then, we have that

$$\lim_{k\to\infty} |||(Y_{\lambda,k}, Z_{\lambda,k}) - (Y_\lambda, Z_\lambda)|||^2 = 0, \quad 0 < t < 1,$$

for every fixed $\lambda \in \Lambda$, where (Y_λ, Z_λ) is the solution of the equation (4) and $(Y_{\lambda,k}, Z_{\lambda,k})$ is the solution of (4) with the terminal condition $\xi_{\lambda,k} \in L^2(\mathbb{R}^m)$.

Main results

Regularity properties

It is known that if

$$\varphi_\lambda(t, Y(t), Z(t)) \xrightarrow{P} \varphi_{\lambda_0}(t, Y(t), Z(t)), \quad \lambda \to \lambda_0 \tag{5}$$

then

$$\lim_{\lambda \to \lambda_0} \int_t^1 \|f_\lambda(s, Y(s), Z(s)) - f_{\lambda_0}(s, Y(s)), Z(s)\|^2 ds = 0. \tag{6}$$

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Main results

Theorem 3.3.

In the hypothesis (i)-(v), we have that if

$$\lim_{\lambda\to\lambda_0} |\xi_\lambda - \xi_{\lambda_0}|^2 = 0,$$

and

$$\lim_{\lambda\to\lambda_0} \int_t^1 \|f_\lambda(s,Y(s),Z(s)) - f_{\lambda_0}(s,Y(s)),Z(s)\|^2 ds = 0$$

then

$$\lim_{\lambda\to\lambda_0} |||(Y_\lambda, Z_\lambda) - (Y_{\lambda_0}, Z_{\lambda_0})|||^2 = 0.$$

Comments

Remark

- For more applications with some unexpected external perturbations, appears a discontinuity in a time moment, (in our paper are two such time moments as to the initial time moment $t = 0$ and to the final time $t = 1$) and before this time moment the stochastic control make out its utility by controlling this perturbations. For example, on more financial markets (in specially for transition financial markets) the strike price for a derivative financial asset is over quoted or higher quoted and this yields a discontinuity in the path of this asset.

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Examples

Example

Example 4.1. We give the following example which sustains the consistency of our condition:

$$f(t,y,z) := e^{\sqrt{t}}(\sqrt{y} - \sqrt[3]{z^2} + 1)$$

and

$$u(t) = \arcsin(t)$$

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