Admissibility and uniform dichotomy for evolution families

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Outline

This presentation is in connection to the field of evolution equations theory, which started to develop in the 1930's. As a starting point for a vast literature concerning this subject, we mention the pioneering work of O. Perron ([13]), who in 1930 was the first to establish the connection between the asymptotic behavior of the solution of the differential equation

$$(A) \qquad \dot{x}(t) = A(t)x(t)$$

and the associated non-homogeneous equation

$$(A,f) \qquad \dot{x}(t) = A(t)x(t) + f(t)$$

in finite dimensional spaces, where A is a $n \times n$ dimensional, continuous and bounded matrix and f is a continuous and bounded function on $\mathbb{R}_+$.

In a paper from 1958 ([8]), J. L. Massera and J. J. Schäffer studied the same problem as O. Perron for differential systems in infinite dimensional spaces. This time they use the pair of spaces (L^p, L^∞), $1 < p < \infty$, where

$$L^p = \{f : [0,\infty) \to X : \int_0^\infty ||f(t)||^p dt < \infty\},$$

$$L^\infty = \{f : [0,\infty) \to X : ess \sup_{t \geq 0} ||f(t)|| < \infty\}.$$

In the monograph [9], the same authors prove that the differential system (A) has uniform dichotomy if and only if the pair of spaces (L^1, L^∞) is admissible to it.

The case of differential systems in finite dimensional spaces was later studied by W. A. Coppel in the monographs [2] (1965) and [3] (1978) and by P. Hartman in [6] (1964).
Further developments for differential systems in infinite dimensional spaces can be found in the monographs of J. L. Daleckij, M. G. Krein [4] (1974) and J. L. Massera, J. J. Schäffer [9] (1966). The case of dynamical systems described by evolution processes was studied by C. Chicone, Y. Latushkin in [1] (1999), by K. J. Engel, R. Nagel in [5] (1999) and by B. M. Levitan, V. V. Zhikov in [7] (1982).

Another important result is the one presented by N. van Minh, F. Räbiger, R. Schnaubelt in [11]. In this paper the authors give a characterization for uniform exponential dichotomy of evolution families with uniform exponential growth, i.e. $||\Phi(t,t_0)|| \leq Me^{\omega(t-t_0)}$ for all $t \geq t_0 \geq 0$, using the pair of spaces $(\mathcal{C}_{00}, \mathcal{C})$, where

$$\mathcal{C} = \{f : [0,\infty) \to X : f \text{ is continuous and bounded}\},$$

$$\mathcal{C}_{00} = \{f \in \mathcal{C} : \lim_{t\to\infty} f(t) = f(0) = 0\}.$$

The admissibility of the above mentioned pair of spaces also implies the existence of a family of projectors compatible with the evolution families used. In order to prove our main result, we use similar techniques as the ones in this paper.

The next important step in studying asymptotic properties of evolution families with uniform exponential growth is the paper [12] of N. van Minh and N. Thieu Huy. The authors use the pair of spaces $(L^p, L^p \cap \mathcal{C})$ and the input-output technique. The admissibility of this pair of spaces implies uniform dichotomy for the evolution families with uniform exponential growth, as well as the existence of a family of projectors compatible with them. Inspired by this paper, we also use this technique, i.e. we choose carefully selected input functions that allow us to prove our main result.

In this presentation we discuss uniform dichotomy of evolution families without any exponential growth, by using the input-output technique and similar techniques found in the above mentioned papers.

Let X be a Banach space and $\mathbb{B}(X)$ the space of all linear and bounded operators acting on X. The norms on X and on $\mathbb{B}(X)$ will be denoted by $||\cdot||$.

Definition

A family of linear operators $\Phi : \Delta = \{(t, t_0) \in \mathbb{R}^2_+ : t \geq t_0 \geq 0\} \to \mathbb{B}(X)$ *on a Banach space* X *is an evolution family if:*

1. $\Phi(t, t) = I$, *for all* $t \in \mathbb{R}_+$, *where* I *is the identity on* X;
2. $\Phi(t, t_0) = \Phi(t, s)\Phi(s, t_0)$, *for all* $t \geq s \geq t_0 \geq 0$;
3. *the map* $\Phi(\cdot, t_0)x$ *is continuous on* $[t_0, \infty)$ *for all* $x \in X$ *and* $\Phi(t, \cdot)x$ *is continuous on* $[0, t]$ *for all* $x \in X$.

Let $X_1(0) = \{x \in X : \Phi(\cdot, 0)x \in L^\infty\}$ be a complemented space in X and let $X_2(0)$ be one of its complements.
We use the spaces

$$L^1(X) = \{f : \mathbb{R}_+ \to X : \int_0^\infty ||f(\tau)||d\tau < \infty\},$$

$$L^\infty(X) = \{f : \mathbb{R}_+ \to X : \ ess \sup_{t \geq 0} ||f(t)|| < \infty\}.$$

The norms on these spaces are

$$||f||_1 = \int_0^\infty ||f(\tau)||d\tau,$$

$$||f||_\infty = ess \sup_{t\geq 0} ||f(t)||.$$

The spaces $(L^1(X), ||\cdot||_1)$ and $(L^\infty(X), ||\cdot||_\infty)$ are Banach spaces.

Definition

Let $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ *be an evolution family. The pair of spaces* $(L^1(X), L^\infty(X))$ *is admissible to* $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ *if and only if for every* f *in* $L^1(X)$ *there exists an element* x *in* X *such that the function*

$$x_f(t) = \Phi(t,0)x + \int_0^t \Phi(t,\tau)f(\tau)d\tau \text{ is in } L^\infty(X).$$

Remark

If the pair $(L^1(X), L^\infty(X))$ is admissible to the evolution family $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$, then for every f in $L^1(X)$ there exists an unique element x in $X_2(0)$ such that the function

$$x_f(t) = \Phi(t,0)x + \int_0^t \Phi(t,\tau)f(\tau)d\tau \text{ is in } L^\infty(X).$$

Remark

If $x \in X_2(0) \setminus \{0\}$, *then* $\Phi(t,0)x \neq 0$, *for all* $t \geq 0$.

Proposition

If $\{\Phi(t,t_0)\}_{t \geq t_0 \geq 0}$ *and the sequence* $(f_n)_n$ *converges to* f *in* $L^1(X)$, *then*

$$\int_0^t \Phi(t,\tau) f_n(\tau) \xrightarrow[n\to\infty]{} \int_0^t \Phi(t,\tau) f(\tau), \ \textit{for all } t \geq 0.$$

Theorem

If the pair $(L^1(X), L^\infty(X))$ *is admissible to the evolution family* $\{\Phi(t, t_0)\}_{t \geq t_0 \geq 0}$*, then there exists* $k > 0$ *such that*

$$||x_f(t)|| \leq k||f||_1 \ a.e. \ t \geq 0,$$

for all $f \in L^1(X)$.

Theorem

The pair $(L^1(X), L^\infty(X))$ is admissible to the evolution family $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ if and only if

(i) *the subspace $X_1(t_0) = \{x \in X : \Phi(\cdot,t_0)x \in L^\infty(X)\}$ is complemented in X and $X_2(t_0) = \Phi(t_0,0)X_2(0)$ is one of its complements, for all $t_0 \geq 0$; also there exists $N > 0$ such that*

$$||\Phi(t,t_0)x|| \leq N||x||,$$

for all $t \geq t_0 \geq 0$ and all $x \in X_1(t_0)$, and

$$N||\Phi(t,t_0)x|| \geq ||x||,$$

for all $t \geq t_0 \geq 0$ and $x \in X_2(t_0)$;

Theorem

(ii) *the function* $\Phi(t_1, t_0) : X_2(t_0) \to X_2(t_1)$ *is invertible for all* $t_1 \geq t_0 \geq 0$;

(iii) *the functions* $t \mapsto ||P_i(t)|| : \mathbb{R}_+ \to \mathbb{R}_+$ *are bounded,* $i = 1, 2$, *where* $P_i(t)$ *are the associated projectors for the subspaces* $X_i(t)$, $i = 1, 2$, *such that:*

$$P_i(t)X = X_i(t),$$

for all $t \geq 0$, *and*

$$P_i(t)\Phi(t, t_0) = \Phi(t, t_0)P_i(t_0),$$

for all $t \geq t_0 \geq 0$, $i = 1, 2$;

(iv) *the functions* $t \mapsto P_i(t)x : \mathbb{R}_+ \to X$ *are continuous,* $i = 1, 2$.

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