# Smoothness of functions and the Weyl-Stone-Titchmarsh-Kodaira theorem

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# Who am I ?

wavelets, function spaces, microlocal analysis,

. . .

# Weyl-Stone-Titchmarsh-Kodaira (cf. [T], [Y])

$$f(x) = \int_{-\infty}^{\infty} \varphi(u) y(x,u)\, d\rho(u),$$

$$\varphi(u) = \int_a^b f(t) y(t,u)\, dt.$$

$\rho$ : density matrix for the operator $L$

$y(x,u)$ : eigenfunction

# WSTK = generalized Fourier

More precisely,

For $L = -(d/dx)^2 + q(x)$,

$$L[y_k(x,\lambda)] = \lambda[y_k(x,\lambda)] \qquad (k = 1, 2),$$

$$^{\exists}\rho_{m,n} \ : \ f(x) = \sum_{m,n=1}^{2} \int_{-\infty}^{\infty} \varphi_n(u) y_m(x,u)\, d\rho_{m,n}(u)$$

Notation. $I_j := (-2^j,\, -2^{j-1}] \cup [2^{j-1},\, 2^j)$,

$$\rho(I_j) := \text{ total variation of } \rho \text{ on } I_j.$$

For $s > 0$, $1 \le q \le \infty$,

$$\|f \mid B_q^s\| := \left[\sum_{j=0}^{\infty} 2^{jsq} \left(\int_{I_j} |\varphi(u)|^2 \, d\rho(u)\right)^{q/2}\right]^{1/q}.$$

$(I_j$ : dyadic interval at $\pm 2^j$.)

**Thm.** If $^\exists \alpha$ : $\rho(I_j) \sim 2^{j\alpha}$ unif. in $j$, then

$$\left[\int_{-\infty}^{\infty} |\varphi(u)|^r |u|^w \, d\rho(u)\right]^{1/r}$$

$$\leq C\|f \mid B_r^s\|, \quad s = \alpha(1/r - 1/2) + w/r > 0.$$

(Proof. Hölder's ineq. cf. [P], [ST])

1914, Bernstein, $B^s_{\infty,\infty}(\mathbb{T})$, $s > 1/2$. [B1, 2]

1946, Szász, $B^{1/2}_{2,1}(\mathbb{T})$. [S]

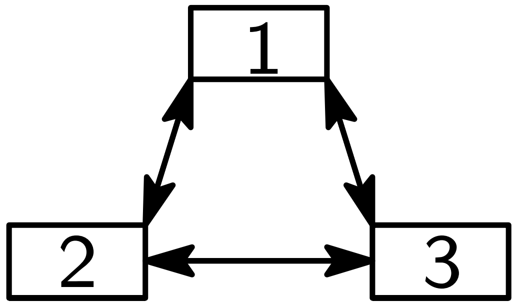
Hardy-Littlewood, Askey, Boas, Wainger, Leindler, Stečkin, Ulyanov, Izumi-Izumi, ...

$a_n \downarrow 0, \quad a_n \geq 0$

$$\left[\sum_{n=1}^{\infty} n^{q(s+1/p')-1} \left(\frac{1}{n} \sum_{m=n/2}^{n} a_m\right)^q\right]^{1/q}$$

$$\leq C \| f \mid B^s_{p,q} \|.$$

Interaction:




1. distribution of eigenvalues for the operator $L$ – "$\rho(I_j) \sim 2^{j\alpha}$" – chapters 7 and 8 of [T], WKB;

2. weighted inequality for $\varphi(u)$ – our thm.;

3. classical theorems as above

wavelet version of WSTK:

$$f(x) = \iint_{\mathbb{R}^2} \varphi(u,v) y(x-v,u)\, d\rho(u,v),$$

$$\varphi(u,v) = \int_a^b f(t) y(t-v,u)\, dt.$$

$$v \ : \ \text{translation}, \quad 1/u \ : \ \text{scaling}$$

$$\text{usually,} \ \ y(t-v,u) = \psi\left((t-v)/u\right)$$

# References.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[B2] S. N. Bernstein, *Sur la convergence absolue des séries trigonométriques* (in Russian), Comm. Soc. Math. Kharkov, 2me série, **14** (1914), 139–144.

[P] J. Peetre, New thoughts on Besov spaces, Duke University Mathematics Series, No. 1. Mathematics Department, Duke University, Durham, 1976.

[S] O. Szász, *On the absolute convergence of trigonometric series*, Ann. of Math. (2) **47** (1946), 213–220.

[ST] H.-J. Schmeisser and H. Triebel, Topics in Fourier analysis and function spaces, A Wiley Interscience Publication, John Wiley & Sons, Ltd., Chichester, 1987.

[T] E. C. Titchmarsh, Eigenfunction expansions associated with second order differential equations, Oxford, 1946.

[Y] K. Yosida, Lectures on differential and integral equations, Translated from the Japanese, Reprint of the 1960 translation, Dover Publications, Inc., New York, 1991.