# Isometries of nonlinear structures of linear operators

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# General problem

**Given a "complex" mathematical structure, how the different "sides" of the structure relate to or interact with each other?**

Martindale's theorem (1969): Any semigroup isomorphism from a ring onto a prime ring which contains a nontrivial idempotent is automatically additive.

### Theorem

**Banach-Stone (1932, 1937)** *Let* $X, Y$ *be compact Hausdorff spaces and* $\phi : C(X) \to C(Y)$ *a surjective linear isometry. Then there exists a homeomorphism* $\varphi : Y \to X$ *and a continuous scalar function* $\tau$ *on* $Y$ *with values of modulus one such that*

$$\phi(f) = \tau \cdot f \circ \varphi \qquad (f \in C(X)).$$

Recall: the transformations $f \mapsto f \circ \varphi$ are exactly the algebra isomorphism between $C(X)$ and $C(Y)$.

Consequence: if a bijection from $C(X)$ onto $C(Y)$ preserves the metric and linear algebraic structures, it necessarily preserves the complete algebraic structure.

$C(X)$: prototype of commutative $C^*$-algebras.

Noncommutative version of Banach-Stone theorem:

**Theorem**

**Kadison (1951)** *Let* $A, B$ *be (unital)* $C^*$*-algebras and* $\phi : A \to B$ *a surjective linear isometry. Then there is a Jordan *-isomorphism* $J : A \to B$ *and a unitary element* $u \in B$ $(u^*u = uu^* = 1)$ *such that* $\phi$ *is of the form*

$$\phi(a) = u \cdot J(a) \qquad (A \in \mathcal{A}).$$

Consequence: Two $C^*$-algebras are linearly isometric iff they are isomorphic as Jordan *-algebras. In fact, surjective linear isometries are closely related to Jordan *-isomorphisms.

Excellent monograph on linear isometries of Banach spaces of functions or operators:

R.J. Fleming and J.E. Jamison, *Isometries on Banach Spaces: Function Spaces,* Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics. 129, Boca Raton, 2003.

R.J. Fleming and J.E. Jamison, *Isometries on Banach Spaces: Vector-valued Function Spaces and Operator Spaces,* Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics. 138, Boca Raton, 2007.

**Question:** How essential is the assumption of linearity?

**Theorem**

**(Mazur-Ulam, 1932)** *Let $X$, $Y$ be normed real linear spaces. Every surjective isometry $\phi : X \to Y$ is affine (preserves the operation of convex combinations) and hence equals a surjective (real-) linear isometry followed by a translation.*

If two normed real linear spaces are "isomorphic" to each other as metric spaces, then they are isomorphic as real linear spaces, too.

One can obtain results saying that if two representatives of the same category of normed algebras are isometric merely as metric spaces, then they are isomorphic as algebras (at least in some sense), too.

# Noncommutative generalization of Mazur-Ulam theorem

Mazur-Ulam theorem gives us that that certain surjective isometries necessarily have algebraic properties. The theorem concerns commutative algebraic structures (normed linear spaces).

**Problem:** Try to find results for the algebraic behavior of isometries on more general algebraic structures, e.g., groups.

**We obtain:** Under certain conditions, the surjective isometries between metric groups (or certain subsets of groups) turn to preserve locally the operation of the inverted Jordan triple product $ba^{-1}b$.

**O. Hatori, G. Hirasawa, T. Miura and L. Molnár, *Isometries and maps compatible with inverted Jordan triple products on groups,* Tokyo J. Math., to appear.**

## Proposition

*Let $G_1, G_2$ be groups equipped with translation and inverse invariant metrics. Let $a, b \in G_1$ for which there exists a constant $K > 1$ such that*

$$d_1(bx^{-1}b, x) \geq Kd_1(x, b)$$

*holds for all $x \in L_{a,b} = \{x \in G_1 : d_1(a, x) = d_1(ba^{-1}b, x) = d_1(a, b)\}$. Then for every surjective isometry $\phi : G_1 \to G_2$ we have*

$$\phi(ba^{-1}b) = \phi(b)\phi(a)^{-1}\phi(b).$$

In the following we suppose that $G_i$ is a group and $X_i$ is a non-empty subset of $G_i$ such that $yx^{-1}y \in X_i$ holds for all $x, y \in X_i$, $i = 1, 2$ (twisted subgroup, Aschbacher).

$X_2$ is said to be 2-divisible if for every $x \in X_2$ there is $y \in X_2$ such that $y^2 = x$.

$X_2$ is called 2-torsion free if the unit element $e_2$ of $G_2$ belongs to $X_2$ and for any $x \in X_2$ the equality $x^2 = e_2$ implies $x = e_2$.

**Proposition**

*Assume $X_i$ is equipped with a metric $d_i$ such that $d_i(cx^{-1}c, cy^{-1}c) = d_i(x, y)$ holds for every triple $x, y, c \in X_i$, $i = 1, 2$. Assume $X_2$ is 2-divisible and 2-torsion free. Let $a, b \in X_1$ be elements for which there exists a constant $K > 1$ such that*

$$d_1(bx^{-1}b, x) \geq K d_1(x, b)$$

*holds for all $x \in L_{a,b} = \{x \in X_1 : d_1(a, x) = d_1(ba^{-1}b, x) = d_1(a, b)\}$. Then for every surjective isometry $\phi : X_1 \to X_2$ we have*

$$\phi(ba^{-1}b) = \phi(b)\phi(a)^{-1}\phi(b).$$

From local preservation of the inverted Jordan triple product to global preservation:

**Lemma**

*For $i = 1, 2$, let $G_i$ be a group and $X_i$ a non-empty subset of $G_i$ such that $yx^{-1}y \in X_i$ holds for every pair $x, y \in X_i$.*
*Suppose that $\phi : X_1 \to X_2$ is a map, $n$ is a positive integer, and $\{a_k\}_{k=0}^{2^n}$ is a finite sequence in $X_1$ such that we have*

$$a_{k+1}a_k^{-1}a_{k+1} = a_{k+2}$$

*and*

$$\phi(a_{k+1}a_k^{-1}a_{k+1}) = \phi(a_{k+1})\phi(a_k)^{-1}\phi(a_{k+1})$$

*for all $0 \le k \le 2^n - 2$. Then it follows that*

$$a_{2^{n-1}}a_0^{-1}a_{2^{n-1}} = a_{2^n}$$

*and*

$$\phi(a_{2^{n-1}}a_0^{-1}a_{2^{n-1}}) = \phi(a_{2^{n-1}})\phi(a_0)^{-1}\phi(a_{2^{n-1}}).$$

With the help of those Mazur-Ulam type general results we have determined the surjective isometries of the unitary group of a Hilbert space.

**O. Hatori and L. Molnár, *Isometries of the unitary group,* PAMS (2012)**.

Here we present new results concerning the surjective isometries of unitary groups in the general setting of $C^*$-algebras.

**Preliminaries:**
Jordan isomorphisms: If $A, B$ are complex algebras, then a linear map $J : A \to B$ is called a Jordan homomorphism if it satisfies $J(a^2) = J(a)^2$ for any $a \in A$ or, equivalently, if it satisfies $J(ab + ba) = J(a)J(b) + J(b)J(a)$ for any $a, b \in A$.

Clearly, every homomorphism $\phi : A \to B$ (that is a linear map such that $\phi(ab) = \phi(a)\phi(b)$ holds for any $a, b \in A$) as well as every antihomomorphism $\psi : A \to B$ (that is a linear map $\psi : A \to B$ satisfying $\psi(ab) = \psi(b)\psi(a)$, $a, b \in A$) is a Jordan homomorphism.

A Jordan *-homomorphism $J : A \to B$ between *-algebras $A, B$ is a Jordan homomorphism which preserves the involution in the sense that it satisfies $J(a^*) = J(a)^*$ for all $a \in A$.

By a Jordan *-isomorphism we mean a bijective Jordan *-homomorphism.

In what follows the units of unital algebras will be denoted by 1.

If $A, B$ are unital algebras and $J : A \to B$ is a surjective Jordan homomorphism, then we have

(i) $J(1) = 1$;

(ii) $J(aba) = J(a)J(b)J(a)$, $a, b \in A$ and this implies that $J(a^n) = J(a)^n$ holds for every $a \in A$ and positive integer $n$;

(iii) for every invertible $a \in A$ we have that $J(a)$ is also invertible and $J(a)^{-1} = J(a^{-1})$.

It then follows that any Jordan isomorphism between unital algebras preserves the spectrum of elements.

If $A, B$ are unital *-algebras and $J : A \to B$ is a surjective Jordan *-homomorphism, then $J$ maps the unitary group of $A$ into the unitary group of $B$. In the case of $C^*$-algebras this easily implies that $J$ is contractive due to the fact that any element of norm less than one is the arithmetic mean of unitaries (Kadison-Pedersen theorem).

If $A$ is a *-algebra, then the real linear subspace of its self-adjoint elements is denoted by $A_s$. If $A$ is unital, by a symmetry in $A$ we mean a self-adjoint unitary element (equivalently, a self-adjoint involution). Clearly, $s \in A$ is a symmetry if and only if it can be written as $s = 2p - 1$ with a projection (self-adjoint idempotent) $p \in A$.

**New results joint with O. Hatori:**

### Theorem

*Let $A_j$ be a unital $C^*$-algebra and $U_j$ be its unitary group, $j = 1, 2$. Assume $\phi : U_1 \longrightarrow U_2$ is a surjective isometry (with respect to the norms given on $A_1, A_2$). Then we have*

$$\phi(\exp(iA_{1s})) = \phi(1) \exp(iA_{2s}) \tag{1}$$

*and there is a central projection $p \in A_2$ and a Jordan *-isomorphism $J : A_1 \longrightarrow A_2$ such that*

$$\phi(\exp(ix)) = \phi(1)(pJ(\exp(ix)) + (1 - p)J(\exp(ix))^*), \quad x \in A_{1s}. \tag{2}$$

### Corollary

*Two unital $C^*$-algebras are isomorphic as Jordan *-algebras if and only if their unitary groups as metric spaces are isometric.*

In the case of general $C^*$-algebras the structure of surjective isometries of unitary groups is given above only on the space of exponentials of skew-symmetric elements. In the case of von Neumann algebras we have the complete description showing thatthe isometries of the unitary groups are closely related to Jordan *-isomorphisms of the von Neumann algebras.

**Corollary**

*Let $M_j$ be a von Neumann algebra and $U_j$ be its unitary group, $j = 1, 2$. A surjective map $\phi : U_1 \to U_2$ is an isometry if and only if there is a central projection $p$ in $M_2$ and a Jordan *-isomorphism $J : M_1 \to M_2$ such that $\phi$ is of the form*

$$\phi(a) = \phi(1)(pJ(a) + (1 - p)J(a)^*), \quad a \in U_1. \tag{3}$$

Cannot be extended to general $C^*$-algebras even not to commutative ones (those are function spaces over compact Hausdorff spaces, the problem appears when the cohomotopy group of the space is nontrivial).

# Thompson isometries of invertible positive elements in $C^*$-algebras

Given a unital $C^*$-algebra $A$, the Thompson metric $d_T$ on the set $A_+^{-1}$ of its invertible positive elements is given by

$$d_T(a,b) = \| \log a^{-1/2} b a^{-1/2} \|, \quad a, b \in A_+^{-1}.$$

Important differential geometrical connections (works by G. Corach etal.): $A_+^{-1}$ is an open subset of the Banach space $A_s$ of all self-adjoint elements of $A$ and hence it is a differentiable manifold which carries a natural Finsler geometrical structure. At any point $a \in A_+^{-1}$, the tangent space is identified with the linear space $A_s$ in which the norm of a vector $x$ is $\|a^{-1/2} x a^{-1/2}\|$.

In the so-obtained Finsler space the geodesic distance is just the Thompson metric $d_T$ on $A_+^{-1}$.

**Molnár, *Thompson isometries of the space of invertible positive operators,* PAMS (2009):** We determined the structure of surjective Thompson isometries of $A_+^{-1}$ in the particular case of the full operator algebra $A = \mathrm{B}(\mathcal{H})$ of all bounded linear operators acting on the complex Hilbert space $H$.

Generalization for unital $C^*$-algebras, new result with O. Hatori:

**Theorem**

*Let $A_j$ be a unital $C^*$-algebra, and $A_{j+}^{-1}$ be the set of all invertible positive elements in $A_j$, $j = 1, 2$. Suppose that $\phi : A_{1+}^{-1} \to A_{2+}^{-1}$ is a surjective map. Then $\phi$ is an isometry with respect to the Thompson metric if and only if there is a central projection $p$ in $A_2$ and a Jordan *-isomorphism $J : A_1 \to A_2$ such that $\phi$ is of the form*

$$\phi(a) = \phi(1)^{1/2}(pJ(a) + (1-p)J(a^{-1}))\phi(1)^{1/2}, \quad a \in A_{1+}^{-1}. \tag{4}$$

In particular, the result says that if the spaces of all invertible positive elements of unital $C^*$-algebras are isometric with respect to the Thompson metric (again, merely as metric spaces), then the underlying algebras are isomorphic as Jordan *-algebras. This means that the metrical - differential geometrical structure of the space of invertible positive elements completely determines the Jordan algebraic structure of the underlying $C^*$-algebra. The result may be viewed as a differential geometry related counterpart of Kadison's theorem.

Let $\mathcal{H}$ be a complex Hilbert space and denote by $\mathrm{P}_1(\mathcal{H})$ the set of all rank-1 projections on $\mathcal{H}$. Let $\|.\|$ stand for the operator norm.

A possible formulation of Wigner's theorem:

**Theorem**

*The bijective transformation $\phi : \mathrm{P}_1(\mathcal{H}) \to \mathrm{P}_1(\mathcal{H})$ satisfies*

$$\|\phi(p) - \phi(q)\| = \|p - q\|, \quad p \in \mathrm{P}_1(\mathcal{H})$$

*if and only if there exists either a unitary or an antiunitary operator $u$ on $\mathcal{H}$ such that*

$$\phi(p) = upu^*, \quad p \in \mathrm{P}_1(\mathcal{H}).$$

Denote by $\mathrm{B}(\mathcal{H})$ the $C^*$-algebra of all bounded operators on $\mathcal{H}$. It is well known that the *-automorphisms of $\mathrm{B}(\mathcal{H})$ are all of the form $a \mapsto uau^*$, for some unitary operator $u$ on $\mathcal{H}$, and its *-antiautomorphisms are all of the form $a \mapsto ua^*u^*$, for some antiunitary operator $u$ on $\mathcal{H}$.

Consequently, the theorem above says that every surjective isometry of $\mathrm{P}_1(\mathcal{H})$ extends either to a *-automorphism or to a *-antiautomorphism of the algebra $\mathrm{B}(\mathcal{H})$.

$\mathrm{P}_1(\mathcal{H})$ is a particular Grassmann space. In fact, one usually defines a Grassmann space as the collection of all subspaces of a Hilbert space with a fixed finite dimension. However, closed subspaces and orthogonal projections (self-adjoint idempotents) are in a one-to-one correspondence. We prefer to consider the spaces $\mathrm{P}_n(\mathcal{H})$ of all projections on $\mathcal{H}$ of rank $n$ ($n$ is a given positive integer) as Grassmann spaces.

The operator norm defines a metric on $\mathrm{P}_n(\mathcal{H})$ which is usually called the gap metric. A lot of applications: perturbation theory of linear operators, perturbation analysis of invariant subspaces, optimization, robust control, multi-variable control, system identification, and signal processing, etc.

**Joint result with F. Botelho and J. Jamison:**

**Theorem**

*Let $\mathcal{H}$ be a complex Hilbert space, $n$ a given positive integer, and $\dim \mathcal{H} \geq 4n$. Assume that the surjective map $\phi : \mathrm{P}_n(\mathcal{H}) \to \mathrm{P}_n(\mathcal{H})$ is an isometry with respect to the gap metric, i.e.,*

$$\|\phi(p) - \phi(q)\| = \|p - q\|, \quad p \in \mathrm{P}_n(\mathcal{H}).$$

*Then there exists either a unitary or an antiunitary operator $u$ on $\mathcal{H}$ such that*

$$\phi(p) = upu^*, \quad p \in \mathrm{P}_n(\mathcal{H}).$$

Consequently, just as in the case of $\mathrm{P}_1(\mathcal{H})$, we obtain that every surjective isometry of the Grassmann space $\mathrm{P}_n(\mathcal{H})$ under the gap metric extends either to a *-automorphism or to a *-antiautomorphism of the full operator algebra $\mathrm{B}(\mathcal{H})$ on $\mathcal{H}$.

A few words about the scheme for the proof. The main ingredient is a non-commutative Mazur-Ulam type result (in a way similar but still different from the ones given above). We deduce that the isometries under consideration preserve the commutativity between the elements of $\mathrm{P}_n(\mathcal{H})$. Next, we give a characterization of orthogonality of rank-$n$ projections involving the relation of commutativity and the gap topology.

**Proposition**

*Let $\mathcal{H}$ be a Hilbert space either infinite dimensional or of finite dimension with $\dim \mathcal{H} \geq 4n$. For any two commuting projections $p, q$ in $\mathrm{P}_n(\mathcal{H})$ we have that $p$ and $q$ are orthogonal if and only if the set $\{p, q\}^c$ as a subspace of the metric space $\mathrm{P}_n(\mathcal{H})$ has a pathwise connected component $K$ such that the maximal number of pairwise commuting projections of rank $n$ in $K^c$ is exactly $\binom{2n}{n}$.*

Using this characterization we show that the orthogonality of the elements of $\mathrm{P}_n(\mathcal{H})$ is preserved under any surjective isometry of $\mathrm{P}_n(\mathcal{H})$. Finally, we complete the proof by applying a nice result due to Győry and Šemrl describing the structure of orthogonality preserving bijections of $\mathrm{P}_n(\mathcal{H})$.

- F. Botelho, J. Jamison and L. Molnár, *Surjective isometries on Grassmann spaces,* preprint.
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Thank you for kind your kind attention!