Uncertainty Principles in Krein Space

by

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- Pauli and Heisenberg (correspondence, 1926-1927)
- Heisenberg Uncertainty Principle (1927)
- E. U. Condon (1929)
- H. P. Robertson (1929)
- J. Von Neumann (1930)
- E. Schrödinger (1930)

1. Formulations

- Heisenberg (1927)

$$\Delta p \Delta q > \frac{h}{2\pi}.$$

- Kennard (1927) and H. P. Robertson (1929)- the case of $L^2(\mathbb{R})$.

  If A, B are self-adjoint, $[A, B] = iC$, $\varphi \in L^2(\mathbb{R})$ then

$$\sigma(A)(\varphi)\sigma(B)(\varphi) > \frac{1}{2}\left|(C\varphi, \varphi)\right|,$$

  where $\sigma(A)(\varphi) = ||(A - \sigma I)\varphi||$ and $\sigma = (A\varphi, \varphi)$.

- The special cases $A\varphi = x\varphi$, $B\varphi = -i\frac{h}{2\pi}\frac{d\varphi}{dx}$ yields Heisenberg's classical inequality.

- Von Neumann (1930)

  $(H, (,))$ a complex Hilbert space, A, B self-adjoint, $[A, B] = aI$ where $a \in \mathbb{C}$, Re $a = 0$. If $||\varphi|| = 1$, then

$$\sigma(A)(\varphi)\sigma(B)(\varphi) > \frac{1}{2}|a|.$$

- Schrödinger (1930) introduces anticommutators as well!

$$[A, B] = AB - BA, \qquad \{A, B\} = AB + BA.$$

Under similar hypo. as Von Neumann above, then

$$\sigma^2(A)(\varphi)\sigma^2(B)(\varphi) \geq$$

$$\frac{1}{4}\left(\left|([A, B]\varphi, \varphi)\right|^2 + \right.$$

$$\left.\left|\{(A - (A\varphi, \varphi)I), (B - (B\varphi, \varphi)I)\}\varphi, \varphi\right|^2\right),$$

Contains all previous versions of the Uncertainty Principle.

Krein Spaces

V a vector sp. over $\mathbb{C}$, $Q : V \times V \to \mathbb{C}$ a sesquilinear hermitian form (linear in its first argument and such that $Q(y, x) = \overline{Q(x, y)}$.)

Writing $[u, v] = Q(u, v)$ we say that a linear transformation A is hermitian if $[Au, v] = [u, Av]$. When the form $[u, u]$ takes on both positive and negative values, the space $(V, [,])$ is called an indefinite inner product space or simply a space with an indefinite metric.

Let $(V, [,])$ be an indefinite inner product space. Then $(V, [,])$ is said to be a Krein Space if

$$V = H^+[+]H^-$$

where $(H^+, +[,])$, $(H^-, -[,])$ are each Hilbert Spaces.

If either H^+ or H^- is finite-dimensional, then V is called a Pontryagin Space.

Example: The vector space $(\mathbb{R}^4, [,])$ where

$$[u, v] = u_1v_1 - u_2v_2 - u_3v_3 - u_4v_4$$

is the Lorentz metric, is a Pontryagin space (also called the Lorentz space).

Basic Results

The canonical decomposition

$$V = H^+[+]H^-$$

induces an inner product $(,)$ via

$$(u, v) = [u^+, v^+] - [u^-, v^-]$$

where $u = u^+ + u^-, v = v^+ + v^-$, $u^\pm \in H^\pm$.

Then $(V, (,))$ is a Hilbert space called the majorant space. The connection between the inner products $[,]$ and $(,)$ is

$$[u, v] = (u^+, v^+) - (u^-, v^-).$$

The orthoprojectors $P_\pm$ defined by $P_\pm(u) = u^\pm$ are such that $P_+ + P_- = I$, and $J = P_+ - P_-$ satisfies $J = J^*$, $JJ^* = I$ and

$$(u, v) = [Ju, v], \quad [u, v] = (Ju, v).$$

Given a Krein space $(V, [,])$, J its fundamental symmetry, $(V, (,))$ its Hilbert majorant space. Let A be s.a. on Krein space V.

For $\varphi \in V$, we define a real (squared) J-standard deviation of A by
$\sigma^2(A)(\varphi) = (A\varphi, A\varphi) - 2(JA\varphi, \varphi)\mathrm{Re}(A\varphi, \varphi) + (JA\varphi, \varphi)^2.$

The Schrödinger principle in Krein space:

Let A, B be s.a. in Krein space V, with fundamental symmetry J, and Hilbert majorant inner product $(,)$. For $\varphi \in V$, $||\varphi|| = 1$, we have

$$\sigma^2(A)(\varphi)\sigma^2(B)(\varphi) \geq \left|(\frac{1}{2}J[A,B]\varphi,\varphi)\right|^2 +$$

$$\{(\frac{1}{2}J\{A,B\}\varphi,\varphi) - (JA\varphi,\varphi)(JB\varphi,\varphi)(2-(J\varphi,\varphi))\}^2,$$

Different *standard deviations* are possible, e.g., If $(V, [,])$ is a Krein space, A, B self-adjoint, $[A, B] = aJ$ where $a \in \mathbb{C}$, Re $a = 0$. If $||\varphi|| = 1$, then

$$\sigma(A)(\varphi)\sigma(B)(\varphi) > \frac{1}{2}\,|a|(J\varphi, \varphi),$$

where now

$$\sigma^2(A)(\varphi) = (A\varphi, A\varphi) - (\mathrm{Re}(A\varphi, \varphi))^2.$$

Extensions to operator dependent commutators (e.g., $[A, B] = a(I + \beta B^2)$) appearing in black-hole entropy in astrophysics can also be formulated.