Exponential stability and uniform boundedness of solutions for nonautonomous periodic abstract Cauchy problems.

Presented by : Dhaou Lassoued
coauthors : Constantin Buşe, Lan Thanh Nguyen, Olivia Saierli

Laboratoire SAMM, Centre P.M.F, Université de Paris I Panthéon-Sorbonne

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Organisation

1. Semigroups and Evolution Families
2. Functional Spaces
3. Evolution semigroup on $E(\mathbb{R}, X)$
4. Main Theorems
5. Lemmas and Differential Equation

Definition-1-

1. A family of bounded linear operators $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ is called a semigroup if :
   $\imath)$ $T(0) = I$.
   $\imath\imath)$ $T(t+s) = T(t) \circ T(s)$, for all $t,\ s \geq 0$.
2. $(T(t))_{t\geq 0}$ is strongly continuous if $\forall x \in X,\ \lim\limits_{t\longrightarrow 0^+} T(t)x = x$.
3. $(T(t))_{t\geq 0}$ is uniformly exponentially stable if $\exists M, \nu$ two positive constants s.t $\|T(t)\| \leq Ne^{-\nu t}$.

Definition-2-

The infinitesimal generator of $(T(t))_{t\geq 0}$ is the operator $(A, D(A))$ defined as :

1. $D(A) := \left\{ u \in X : \lim\limits_{t\to 0^+} \dfrac{T(t)u - u}{t} \text{ exists} \right\}$.
2. $Au := \lim\limits_{t\to 0^+} \dfrac{T(t)u - u}{t}$, for all $u \in D(A)$.

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Definition-3-

A family of linear operators $\mathcal{U} = \{U(t, s) : t \geq s\} \subset \mathcal{L}(X)$ is called evolution family on X if :

1. $U(t, s)U(s, r) = U(t, r)$ for all $t \geq s \geq r \geq 0$
2. $U(t, t) = I$ for $t \geq 0$.

Definition-4-

1. An evolution family $\mathcal{U}$ on X is called strongly continuous if for each $x \in X$, the map

$$(t,s) \mapsto U(t,s)x : \{(t,s) \in \mathbb{R}^2 : t \geq s \geq 0\} \to X$$

is continuous.

2. An evolution family $\mathcal{U}$ has exponential growth if there exist the constants $\widetilde{M} \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|U(t,s)\| \leq \widetilde{M} e^{\omega(t-s)}, \text{ for all } t \geq s.$$

3. An evolution family $\mathcal{U}$ is q-periodic, for some positive q, if

$$U(t+q, s+q) = U(t,s) \text{ for all pairs } (t,s) \text{ with } t \geq s \geq 0.$$

Every strongly continuous and q-periodic evolution family acting on a Banach space has an exponential growth.

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Definition-5-

1. $BUC(\mathbb{R}, X)$ is the space of all X-valued bounded uniformly continuous functions defined on $\mathbb{R}$, endowed with the "sup" norm $\|f\|_\infty := \sup_{t\in\mathbb{R}} \|f(t)\|$.
2. $P_q(\mathbb{R}, X)$ is the subspace of $BUC(\mathbb{R}, X)$ consisting of all functions F such that $F(t+q) = F(t)$ for all $t \in \mathbb{R}$.
3. $AP_1(\mathbb{R}, X)$ is the space of all X-valued functions defined on $\mathbb{R}$, which can be represented as $f(t) = \sum_{k=-\infty}^{k=\infty} e^{i\mu_k t} c_k(f)$ for all $t \in \mathbb{R}$, where $\mu_k \in \mathbb{R}$, $c_k(f) \in X$ and $\|f\|_1 := \sum_{k=-\infty}^{k=\infty} \|c_k(f)\| < \infty$.
4. Evidently, the spaces $BUC(\mathbb{R}_+, X)$, $P_q(\mathbb{R}_+, X)$ and $AP_1(\mathbb{R}_+, X)$ may be defined in a similar manner.

Frame of the result

For an arbitrary $t \geq 0$, we denote by $\mathcal{A}_t$ the set of all X-valued functions f defined on $\mathbb{R}$ for which there exists a function $F \in P_q(\mathbb{R}, X) \cap AP_1(\mathbb{R}, X)$ such that $F(t) = 0$, $f = F_{|_{[t,\infty)}}$ and $f(s) = 0$ for all $s < t$.

We define $E(\mathbb{R}, X) := \overline{\text{span}}\, \{\cup_{t \geq 0} \mathcal{A}_t\}$ which is a closed subspace of $BUC(\mathbb{R}, X)$ endowed with the "sup"-norm.

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Evolution semigroup on $E(\mathbb{R}, X)$

We define the evolution semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$, associated to a strongly continuous and q-periodic evolution family $\mathcal{U}$, on $E(\mathbb{R}, X)$ by

$$(\mathcal{T}(t)f)(s) := \begin{cases} U(s, s-t)f(s-t), & s \geq t \\ 0, & s < t, \end{cases} \qquad t \geq 0, s \in \mathbb{R}. \tag{1}$$

1. The evolution semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ defined in (1) acts on $E(\mathbb{R}, X)$.
2. The evolution semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ is strongly continuous.

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Theorem-1-

Let $\mathcal{U} = \{U(t,s) : t \geq s\}$ be a strongly continuous and q-periodic evolution family on X. If

$$\sup_{\mu \in \mathbb{R}} \sup_{t \geq \tau \geq 0} \left\| \int_\tau^t e^{i\mu s} U(t,s)x ds \right\| := M(x) < \infty, \quad \forall x \in X,$$

then,

$$\sup_{\mu \in \mathbb{R}} \sup_{t \geq 0} \left\| \int_0^t e^{-i\mu s} \mathcal{T}(s) f ds \right\| \leq L(f) < \infty, \quad \forall f \in \text{span}\{\cup_{t \geq 0} \mathcal{A}_t\}.$$

In addition, if for each $x \in X$, the map $s \mapsto U(s,0)x$ satisfies a Lipschitz condition on $(0,q)$, then the family $\mathcal{U}$ is uniformly exponentially stable.

Theorem-2-

Let $\mathcal{U}$ be a strongly continuous and q-periodic evolution family on a Banach space X and let $\mathcal{T}$ be the evolution semigroup associated to the family $\mathcal{U}$ on $E(\mathbb{R}, X)$. Denote by G its infinitesimal generator. Consider the statements :

- **(1)** $\mathcal{U}$ is uniformly exponentially stable.
- **(2)** $\mathcal{T}$ is uniformly exponentially stable.
- **(3)** G is invertible.
- **(4)** For each $f \in E(\mathbb{R}, X)$ the map

  $$t \mapsto g_f(t) := \int_0^t U(t,s)f(s)ds$$

  belongs to $E(\mathbb{R}, X)$.
- **(5)** For each $f \in E(\mathbb{R}, X)$ the map g_f belongs to $BUC(\mathbb{R}_+, X)$.

Then,

$$\textbf{(1)} \Leftrightarrow \textbf{(2)} \Rightarrow \textbf{(3)} \Rightarrow \textbf{(4)} \Rightarrow \textbf{(5)}.$$

If, in addition, the family $\mathcal{U}$ is uniformly bounded, then $(\mathbf{5}) \Rightarrow (\mathbf{3})$.

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Then,

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If, in addition, the family $\mathcal{U}$ is uniformly bounded, then $(\mathbf{5}) \Rightarrow (\mathbf{3})$.

The proof of these two theorems uses the foloowing lemmas :

Lemma 1, [Buşe, Cerone, Dragomir, Sofo, 2005]

If

$$\sup_{n\geq 1}\left\|\sum_{k=0}^{n} e^{-i\mu k}U(q,0)^k\right\| := M(\mu) < \infty \tag{2}$$

then $e^{i\mu}$ belongs to the resolvent set of $U(q,0)$. Moreover, if (2) holds for every $\mu \in \mathbb{R}$ then $r(U(q,0)) < 1$, i.e. the family $\mathcal{U}$ is uniformly exponentially stable.

Lemma 2 [Reghiş, Buşe, 1998], [Phong, 2001]

Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup generated by A on a Banach space X. If for a given real μ, one has $\sup_{t\geq 0}\left\|\int_0^t e^{-i\mu s}T(s)x\,ds\right\| < \infty, \quad \forall x \in X$, then μi belongs to $\rho(A)$.

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Lemma 3

Let $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ be a strongly continuous and q-periodic evolution family acting on X. Assume that $\mathcal{U}$ is uniformly bounded, i.e. $\sup_{t\geq s\geq 0} ||U(t,s)|| := K < \infty$. If for a real number μ and a function f in $E(\mathbb{R}, X)$, have that

$$\sup_{t\geq 0}\left\|\int_0^t e^{i\mu s}U(t,s)f(s)ds\right\| := M(\mu, f) < \infty,$$

then,

$$\sup_{t\geq 0}\left\|\int_0^t e^{-i\mu s}\mathcal{T}(s)fds\right\| \leq K(\mu, f) < \infty.$$

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Lemma 4 [Minh, Räbiger, Schnaubelt, 1998]

Let $f, u \in E(\mathbb{R}, X)$. The following two statements are equivalent :

- $u \in D(G)$ and $Gu = -f$.
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In terms of nonautonomous periodic abstract Cauchy problems

$$\begin{cases} \dot{u}(t) = A(t)u(t) + e^{i\mu t}x, & t > s \\ u(s) = 0, \end{cases}$$

Differential Equation

Let $(A(t), D(A(t)))_{t\geq 0}$ be a family of linear operators acting on a Banach space X. Assume that the evolution family $\mathcal{U} = \{U(t,s) : t \geq s \geq 0\}$, generated by $\{A(t)\}_{t\geq 0}$, verifies all assumptions in Theorem-1-. Then, $\mathcal{U}$ is uniformly exponentially stable if and only if for each $\mu \in \mathbb{R}$ and each $s \geq 0$, the solution of the abstract Cauchy Problem

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Thank You