Spectral mapping theorems for contractions

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Absolutely continuous contractions

$\mathcal{H}$ complex, separable Hilbert space, $\dim \mathcal{H} = \aleph_0$

$\mathcal{L}(\mathcal{H})$ bounded, linear operators on $\mathcal{H}$

Definition

$T \in \mathcal{L}(\mathcal{H})$ is an *absolutely continuous contraction*, if $\|T\| \leq 1$,

and $T = T_a \oplus T_c$, where T_a is a.c. unitary and T_c is c.n.u. contraction.

Unitary dilation

$T \in \mathcal{L}(\mathcal{H})$ a.c. contraction

$U \in \mathcal{L}(\mathcal{G})$ *minimal unitary dilation* of T (Sz.-Nagy):

(i) $\mathcal{H} \subset \mathcal{G}$, $\vee_{n=-\infty}^{\infty} U^n \mathcal{H} = \mathcal{G}$,

(ii) $T^n = P_{\mathcal{H}} U^n | \mathcal{H} \ \forall n \in \mathbb{Z}_+$.

U a.c. unitary operator (Sz.-Nagy – Foias)

$L^\infty = L^\infty(\mu)$ abelian C^*-algebra

μ normalized Lebesgue measure on the unit circle $\mathbb{T}$

$\chi(\zeta) = \zeta \ \forall \zeta \in \mathbb{T}$

$\exists! \ \Phi_U \colon L^\infty \to \mathcal{L}(\mathcal{G}), f \mapsto f(U)$ weak-$*$ continuous, contractive,

unital algebra-homomorphism, $\chi \mapsto U$ (Spectral Theorem)

Hardy class H^∞

$$H^\infty = \left\{ f \in L^\infty : \widehat{f}(-n) = \int_{\mathbb{T}} f \chi^n \, d\mu = 0 \ \ \forall n \in \mathbb{N} \right\}$$

Hardy space, weak-$*$-closed subalgebra of L^∞

$$f \in H^\infty \implies F(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|1-\overline{\zeta} z|^2} f(\zeta) \, d\mu(\zeta) \quad (z \in \mathbb{D})$$

bounded analytic function on $\mathbb{D}$

$F \colon \mathbb{D} \to \mathbb{C}$ bounded analytic $\implies$ $f \in H^\infty$, where

$$f(\zeta) = \lim_{r \to 1-0} F(r\zeta) \ \text{ for a.e. } \zeta \in \mathbb{T}$$

Notation: $f \equiv F$

Functional calculus

$\Phi_T : H^\infty \to \mathcal{L}(\mathcal{H}),\ f \mapsto f(T) := P_{\mathcal{H}} f(U)|\mathcal{H}$

(i) weak-$*$ continuous,

(ii) contractive,

(iii) unital algebra-homomorphism,

(iv) $\chi \mapsto T$.

Uniquely determined

Sz.-Nagy–Foias functional calculus for T

Theorem (Foias – Mlak, special version)

If $f \in H^\infty$ can be continuously extended to the closed unit disc $\mathbb{D}^-$, then

$$\sigma(f(T)) = f(\sigma(T)).$$

The positive operator A_T induced by T

$T \in \mathcal{L}(\mathcal{H})$ a.c. contraction

$T^{*n}T^n \to A_T$ in SOT, $0 \leq A_T \leq I$, $\mathcal{K}_T^+ := (A_T\mathcal{H})^-$

$$X_T^+ : \mathcal{H} \to \mathcal{K}_T^+, \ x \mapsto A_T^{1/2}x$$

$T^*A_TT = A_T \implies \|X_T^+Tx\| = \|X_T^+x\| \ \forall x \in \mathcal{H}$

$\implies \exists! V_T^+ \in \mathcal{L}(\mathcal{K}_T^+)$ isometry, $X_T^+T = V_T^+X_T^+$

$V_T \in \mathcal{L}(\mathcal{K}_T)$ minimal unitary extension of V_T^+, a.c. unitary

$$X_T : \mathcal{H} \to \mathcal{K}_T, \ x \mapsto X_T^+x; \quad X_TT = V_TX_T$$

Unitary asymptote

Definition

(X, V) is a *unitary asymptote* of the contraction T, if

(i) $V \in \mathcal{L}(\mathcal{K})$ unitary,

(ii) $X \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ contractive, $XT = VX$,

(iii) $\forall (X', V')$ satisfying (i) and (ii), $\exists! Y \in \mathcal{L}(\mathcal{K}, \mathcal{K}')$ contractive, $YV = V'Y$ and $X' = YX$.

Existence: (X_T, V_T) is a realization

Uniqueness: up to isomorphism

$\ker(X) = \{x \in \mathcal{H} : \lim_n \|T^n x\| = 0\} =: \mathcal{H}_0(T)$ hyperinvariant subspace

of stable vectors for T

Residual set

$T \in \mathcal{L}(\mathcal{H})$ a.c. contraction

(X, V) unitary asymptote of T

E spectral measure of the a.c. unitary operator $V \in \mathcal{L}(\mathcal{K})$

Definition

The measurable support of E is called the *residual set* of T, denoted by $\omega(T)$.

Uniquely determined up to sets of zero Lebesgue measure.

(Unique assuming that each point is of full density.)

$$\omega(T) \subset \sigma(V) \subset \sigma(T)$$

Partially inner functions

Definition

$f \in H^\infty$ is a *partially inner function*, if

(i) $|f(0)| < 1 = \|f\|_\infty$, and

(ii) $\Omega(f) := \{\zeta \in \mathbb{T} : |f(\zeta)| = 1\}$ is of positive measure.

f is *regular*, if $\alpha \subset \Omega(f),\ \mu(\alpha) = 0 \implies \mu(f(\alpha)) = 0$,

or equivalently, if $f(\alpha)$ is measurable, whenever $\alpha \subset \Omega(f)$ is measurable

Spectral mapping theorem for the residual set

$T \in \mathcal{L}(\mathcal{H})$ a.c. contraction

(X, V) unitary asymptote of T, E spectral measure of $V \in \mathcal{L}(\mathcal{K})$

$f \in H^\infty$ partially inner function

Then $f(T) \in \mathcal{L}(\mathcal{K})$ is an a.c. contraction.

$$\widetilde{\mathcal{K}} := E(\omega(T) \cap \Omega(f))\mathcal{K} \in \text{Hlat } V, \quad \widetilde{V} := V|\widetilde{\mathcal{K}}$$

$$\widetilde{X} \colon \mathcal{H} \to \widetilde{\mathcal{K}}, \ x \mapsto E(\omega(T) \cap \Omega(f))Xx$$

Theorem

(a) *$(\widetilde{X}, f(\widetilde{V}))$ is a unitary asymptote of $f(T)$.*

(b) *If f is regular, then $\omega(f(T)) = f\,(\omega(T) \cap \Omega(f))$.*

Partial orderings

Definition

For $f, g \in H^\infty$, $f \prec g$ if $|f(z)| \le |g(z)|$ $\forall z \in \mathbb{D}$.

Reflexive, transitive; $f \prec g$ and $g \prec f \implies g = \kappa f$ for some $\kappa \in \mathbb{T}$.

Definition

For $A, B \in \mathcal{L}(\mathcal{H})$, $A \prec B$ if $\|Ax\| \le \|Bx\|$ $\forall x \in \mathcal{H}$.

Reflexive, transitive; $A \prec B$ and $B \prec A \implies B = ZA$ with a partial isometry Z.

$$f \prec g \implies \exists \eta \in H^\infty,\ \|\eta\|_\infty \le 1,\ f = \eta g \implies f(T) = \eta(T)g(T) \prec g(T)$$

Proposition

Φ_T is monotone.

Decreasing sequences

$F = \{f_n\}_{n=1}^{\infty}$ decreasing sequence in H^∞ : $f_{n+1} \prec f_n \ \forall n \in \mathbb{N}$

$\varphi_F(\zeta) := \lim_{n\to\infty} |f_n(\zeta)|$ for a.e. $\zeta \in \mathbb{T}$

bounded, measurable limit function on $\mathbb{T}$

$$\mathcal{H}_0(T, F) := \left\{ x \in \mathcal{H} : \lim_{n\to\infty} \|f_n(T)x\| = 0 \right\} \in \text{Hlat } T$$

$\mathcal{B}(\mathbb{T})$: Borel subsets of $\mathbb{T}$

For $\alpha \in \mathcal{B}(\mathbb{T})$, $\mathcal{N}(F, \alpha) := \{\zeta \in \alpha : \varphi_F(\zeta) > 0\}$

Contractions asymptotically non-vanishing on a set

Definition

T is (asymptotically) *non-vanishing on* $\alpha \in \mathcal{B}(\mathbb{T})$, if $\mathcal{H}_0(T, F) = \{0\}$,

whenever φ_F is non-vanishing on α : $\mu(\mathcal{N}(F, \alpha)) > 0$.

$\mathcal{NV}(T)$: system of Borel sets, where T is non-vanishing

$$\delta_T = \sup \{\mu(\alpha) : \alpha \in \mathcal{NV}(T)\}$$

Choosing $\{\alpha_n\}_{n=1}^{\infty} \subset \mathcal{NV}(T)$ with $\lim_{n\to\infty} \mu(\alpha_n) = \delta_T$, we obtain that

$\widetilde{\alpha} = \cup_{n=1}^{\infty} \alpha_n$ is maximal in $\mathcal{NV}(T)$: $\alpha \subset \widetilde{\alpha}$ $(\mu(\alpha \setminus \widetilde{\alpha}) = 0)$ $\forall \alpha \in \mathcal{NV}(T)$.

Quasianalytic spectral set

Definition

The *quasianalytic spectral set* $\pi(T)$ of T is the largest set in $\mathcal{N}\mathcal{V}(T)$.

$\pi(T)$ is uniquely determined up to sets of zero measure

(Unique, assuming that its points are of full density.)

$$\mu(\pi(T)) > 0 \implies \mathcal{H}_0(T) = \mathcal{H}_0\left(T, \{\chi^n\}_n\right) = \{0\} \implies T \text{ is a } C_{1\cdot}\text{-contraction}$$

Spectral mapping theorem for the quasianalytic spectral set

$T \in \mathcal{L}(\mathcal{H})$ a.c. contraction

$f \in H^\infty$ partially inner, $\Omega(f) := \{\zeta \in \mathbb{T} : |f(\zeta)| = 1\}$

Theorem

If f is regular, then

$$\pi(f(T)) \supset f\,(\pi(T) \cap \Omega(f))\,.$$

Quasianalytic contractions

Theorem

(a) $\mu(\pi(T) \setminus \omega(T)) = 0$ *always holds.*

(b) *If* $\mu(\omega(T) \setminus \pi(T)) > 0$, *then Hlat* T *is non-trivial.*

Definition

T is a *quasianalytic contraction*, if $\pi(T) = \omega(T)$.

Thus (HSP) is answered affirmatively for *non-quasianalytic contractions*.

Construction of quasianalytic contractions

Theorem

If $f \in H^\infty$ *is a regular partially inner function and*

$$\omega(T) \cap \Omega(f) = \pi(T) \cap \Omega(f),$$

then $f(T)$ *is a quasianalytic contraction with*

$$\pi(f(T)) = f\left(\pi(T) \cap \Omega(f)\right).$$

Proposition

(a) *If* T_1, T_2 *are quasianalytic and* $\pi(T_1) = \pi(T_2)$, *then* $T = T_1 \oplus T_2$ *is quasianalytic with* $\pi(T) = \pi(T_1)$.

(b) *If* T *is quasianalytic and* $\mathcal{M}$ *is a non-zero invariant subspace of* T, *then* $T|\mathcal{M}$ *is quasianalytic with* $\pi(T|\mathcal{M}) = \pi(T)$.

Examples

(a) $S \in \mathcal{L}(H^2)$, $Sf = \chi f$ simple unilateral shift is quasianalytic because of the F. & M. Riesz Theorem.

(b) $T_\psi \in \mathcal{L}(H^2)$, $T_\psi f = \psi f$ analytic Toeplitz operator is quasianalytic, provided $\psi \in H^\infty$ is a partially inner function, with $\alpha = \Omega(\psi)$.
The natural realization of the unitary asymptote of T_ψ is $(X_\alpha, M_{\alpha,\psi})$, where

$$X_\alpha \colon H^2 \to L^2(\alpha), \quad f \mapsto \chi_\alpha f,$$

$$M_{\alpha,\psi} \colon L^2(\alpha) \to L^2(\alpha), \quad g \mapsto \psi g.$$

(Here $L^2(\alpha) = \chi_\alpha L^2(\mathbb{T})$, where χ_α stands for the characteristic function of α.)

(c) Every C_{10}-contraction T with $\dim(I - T^*T)\mathcal{H} < \infty$ is quasianalytic.

Cyclic contractions

$T \in \mathcal{L}(\mathcal{H})$ a.c. contraction is *cyclic*, if

$$\exists\, v \in \mathcal{H}, \quad \vee_{n=0}^{\infty} T^n v = \mathcal{H}.$$

Then the unitary asymptote of T can be chosen in the form (X, M_α), where $\alpha = \omega(T) \subset \mathbb{T}$ Borel set and

$$M_\alpha \in \mathcal{L}(L^2(\alpha)), \quad M_\alpha f = \chi f.$$

The commutant of M_α:

$$\{M_\alpha\}' := \left\{ C \in \mathcal{L}(L^2(\alpha)) : CM_\alpha = M_\alpha C \right\} = \{M_{\alpha,\psi} : \psi \in L^\infty(\alpha)\}.$$

Spectral mapping theorem for cyclic contractions

Theorem (K – Totik)

T is a cyclic, quasianalytic, a.c. contraction.

(a) *There exists a conformal mapping $f\colon \mathbb{D} \to \mathbb{D}$, which is a regular partially inner function, satisfying the condition*

$$f(\Omega(f) \cap \alpha) = \mathbb{T}.$$

(b) *The operator $f(T)$ is a cyclic, quasianalytic, a.c. contraction, satisfying the conditions*

   (i) $\pi(f(T)) = \mathbb{T}$,

   (ii) $\{f(T)\}' = \{T\}'$, *and so Hlat $f(T)$ = Hlat T.*

Proposition

There exists an invariant subspace $\mathcal{M}$ of $f(T)$ such that $f(T)|\mathcal{M}$ is similar to the unilateral shift S.

Actually, the subspaces of this kind span the whole space $\mathcal{H}$.

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