Abstract boundary mappings; recent developments and some history

Seppo Hassi

Based on two joint papers with

(1) Vladimir Derkach, Mark Malamud, and Henk de Snoo
and
(2) with Rudi Wietsma

Timisoara, October 12, 2012

Boundary relations and Weyl families

Basic notations

$\mathfrak{H}$ a Hilbert space with inner product $(\cdot,\cdot)$.

S a closed symmetric linear relation in $\mathfrak{H}$ with arbitrary defect numbers.

• The following definitions and facts are taken from [DHMS2006]:

Definition

With $\mathcal{H}$ a Hilbert space a linear relation $\Gamma : \mathfrak{H}^2 \mapsto \mathcal{H}^2$ is a *boundary relation* for S^*, if:

(G1) $\operatorname{dom}\Gamma$ is dense in S^* and

$$(f',g)_{\mathfrak{H}} - (f,g')_{\mathfrak{H}} = (h',k)_{\mathcal{H}} - (h,k')_{\mathcal{H}}, \tag{1.1}$$

holds for every $\{\widehat{f},\widehat{h}\}$, $\{\widehat{g},\widehat{k}\} \in \Gamma$;

(G2) Γ is maximal in the sense that if $\{\widehat{g},\widehat{k}\} \in \mathfrak{H}^2 \times \mathcal{H}^2$ satisfies (1.1) for every $\{\widehat{f},\widehat{h}\} \in \Gamma$, then $\{\widehat{g},\widehat{k}\} \in \Gamma$.

The condition (G1) can be interpreted as an abstract *Green's identity*.

Associate with Γ the following linear relations which are not necessarily closed:

$$\begin{aligned} \Gamma_0 &= \left\{ \{\widehat{f},h\} : \{\widehat{f},\widehat{h}\} \in \Gamma,\ \widehat{h} = \{h,h'\} \right\}, \\ \Gamma_1 &= \left\{ \{\widehat{f},h'\} : \{\widehat{f},\widehat{h}\} \in \Gamma,\ \widehat{h} = \{h,h'\} \right\}. \end{aligned} \tag{1.2}$$

Boundary relations as unitary mappings between Krein spaces

Consider $(\mathfrak{H}^2, J_{\mathfrak{H}})$ as a Kreĭn space with scalar product

$$\left[\begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} g \\ g' \end{pmatrix}\right]_J := i(f', g) - i(f, g')$$

determined on $\mathfrak{H}^2 = \mathfrak{H} \times \mathfrak{H}$ by $J_{\mathfrak{H}} := \begin{pmatrix} 0 & -iI_{\mathfrak{H}} \\ iI_{\mathfrak{H}} & 0 \end{pmatrix}$.

Now the condition (G1) can be interpreted as follows:
Γ is an *isometric multivalued* mapping from the Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$:

$$(J_{\mathfrak{H}}\widehat{f}, \widehat{g})_{\mathfrak{H}^2} = (J_{\mathcal{H}}\widehat{h}, \widehat{k})_{\mathcal{H}^2}, \quad \{\widehat{f}, \widehat{h}\}, \{\widehat{g}, \widehat{k}\} \in \Gamma.$$

The maximality condition (G2) guarantees that a boundary relation Γ is a *unitary relation* from $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to $(\mathcal{H}^2, J_{\mathcal{H}})$:

$$\Gamma^{-1} = \Gamma^{[*]}.$$

In particular, Γ is closed and linear.
Converse is also true:

Proposition

Let Γ be a unitary relation from the Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$. Then:

$$\Gamma \text{ boundary relation for } S^* \quad \Longleftrightarrow \quad \ker \Gamma = S.$$

Weyl families and γ-fields

Denote

$$\mathfrak{N}_\lambda(T) = \ker\,(T-\lambda), \qquad \widehat{\mathfrak{N}}_\lambda(T) = \{\,\{f,\lambda f\} \in T : \, f \in \mathfrak{N}_\lambda(T)\,\}.$$

Definition

The *Weyl family* $M(\cdot)$ of S corresponding to the boundary relation $\Gamma : \mathfrak{H}^2 \mapsto \mathcal{H}^2$ is defined by $M(\lambda) := \Gamma(\widehat{\mathfrak{N}}_\lambda(T))$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, i.e.,

$$M(\lambda) = \left\{ \widehat{h} \in \mathcal{H}^2 : \, \{\widehat{f}_\lambda, \widehat{h}\} \in \Gamma, \, \widehat{f}_\lambda = \{f, \lambda f\} \in \mathfrak{H}^2 \right\},$$

If $M(\cdot)$ is operator-valued, then it is called the *Weyl function* of S corresponding to Γ.

Definition

The γ-*field* $\gamma(\cdot)$ of S corresponding to the boundary relation $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ is defined by

$$\gamma(\lambda) := \left\{ \{h, f\} \in \mathcal{H} \times \mathfrak{H} : \, \{\widehat{f}_\lambda, \widehat{h}\} \in \Gamma, \; \widehat{f}_\lambda \in \mathfrak{H}^2 \right\},$$

where $\widehat{f}_\lambda = \{f, \lambda f\}$, $\widehat{h} = \{h, h'\}$, and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Moreover, $\widehat{\gamma}(\lambda)$ stands for

$$\widehat{\gamma}(\lambda) := \left\{ \{h, \widehat{f}_\lambda\} \in \mathcal{H} \times \mathfrak{H}^2 : \{h, f\} \in \gamma(\lambda) \right\}.$$

γ-field is a single-valued mapping from $\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \mathrm{dom}\, M(\lambda)$ onto $\mathfrak{N}_\lambda(T)$, $T = \mathrm{dom}\,\Gamma$ (T dense in S^*).

Nevanlinna families

A family of linear relations $M(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, in a Hilbert space $\mathcal{H}$ is called a *Nevanlinna family* if:

(i) $M(\lambda)$ is maximal dissipative for every $\lambda \in \mathbb{C}_+$ (resp. max. accumulative for every $\lambda \in \mathbb{C}_-$);
(ii) $M(\lambda)^* = M(\bar{\lambda})$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$;
(iii) for some, and hence for all, $\mu \in \mathbb{C}_+(\mathbb{C}_-)$ the operator family $(M(\lambda) + \mu)^{-1} (\in [\mathcal{H}])$ is holomorphic for all $\lambda \in \mathbb{C}_+(\mathbb{C}_-)$.

By the maximality condition, $M(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, is closed. The *class of all Nevanlinna families* in a Hilbert space is denoted by $\widetilde{R}(\mathcal{H})$.

Nevanlinna families $M(\lambda) \in \widetilde{R}(\mathcal{H})$ admit the following decomposition to the operator part $M_s(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and constant multi-valued part M_∞:

$$M(\lambda) = M_s(\lambda) \oplus M_\infty, \quad M_\infty = \{0\} \times \mathrm{mul}\, M(\lambda).$$

Here $M_s(\lambda)$ is a Nevanlinna family of densely defined operators in $\mathcal{H} \ominus \mathrm{mul}\, M(\lambda)$.

Realization theorem for Nevanlinna families

Two boundary relations $\Gamma^{(j)} : (\mathfrak{H}^{(j)})^2 \to \mathcal{H}^2$, $j = 1, 2$, are said to be *unitarily equivalent* if there is a unitary operator $U : \mathfrak{H}^{(1)} \to \mathfrak{H}^{(2)}$ such that

$$\Gamma^{(2)} = \left\{ \left\{ \begin{pmatrix} Uf \\ Uf' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} : \left\{ \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Gamma^{(1)} \right\}. \tag{1.3}$$

If the boundary relations $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are connected by (1.3) and $S_j = \ker \Gamma^{(j)}$, $T_j = \mathrm{dom}\, \Gamma^{(j)}$, $j = 1, 2$, then

$$S_2 = US_1U^{-1}, \quad T_2 = UT_1U^{-1}.$$

The boundary relation $\Gamma : \mathfrak{H}^2 \mapsto \mathcal{H}^2$ is *minimal*, if

$$\mathfrak{H} = \mathfrak{H}_{min} := \overline{\mathrm{span}}\, \{ \mathfrak{N}_\lambda(T) : \lambda \in \mathbb{C}_+ \cup \mathbb{C}_- \}.$$

Theorem (DHMS2006)

Let $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ be a boundary relation for S^*. Then the corresponding Weyl family $M(\cdot)$ belongs to the class $\widetilde{R}(\mathcal{H})$.
Conversely, if $M(\cdot)$ belongs to the class $\widetilde{R}(\mathcal{H})$ then there exists (up to unitary equivalence) a unique minimal boundary relation whose Weyl function coincides with $M(\cdot)$.

Unitary relations in Krein spaces

Recall that a linear relation $T : (\mathfrak{H}_1, J_1) \to (\mathfrak{H}_2, J_2)$ is *unitary* if

$$T^{-1} = T^{[*]}. \tag{1.4}$$

A unitary relation is automatically closed. The definition (1.4) and the following proposition go back to Shmulyan 1976; see [DHMS06].

Proposition

Let T be a unitary relation from $(\mathfrak{H}_1, J_1)$ to $(\mathfrak{H}_2, J_2)$. Then

$$\mathrm{dom}\, T = \mathrm{ran}\, T^{[*]}, \quad \mathrm{ran}\, T = \mathrm{dom}\, T^{[*]},$$

and

$$\ker\, T = (\mathrm{dom}\, T)^{[\perp]}, \quad \mathrm{mul}\, T = (\mathrm{ran}\, T)^{[\perp]}.$$

Moreover, $\mathrm{dom}\, T$ *is closed if and only if* $\mathrm{ran}\, T$ *is closed.*

A unitary relation T is an operator if and only if $\overline{\mathrm{ran}}\, T = \mathfrak{H}$.

Corollary

Let T be a unitary relation from $(\mathfrak{H}_1, J_1)$ to $(\mathfrak{H}_2, J_2)$. Then the following statements are equivalent:

(i) $\mathrm{ran}\, T = \mathfrak{H}$;

(ii) *T is a bounded linear operator (with* $\mathrm{dom}\, T = (\ker\, T)^{[\perp]}$*).*

Unitary boundary triplets of bounded type

The notion of the ordinary boundary triplet was extended by Derkach and Malamud in 1995 by weakening the surjectivity assumption on Γ.

Definition

Let S be a closed symmetric operator in a Hilbert space $\mathfrak{H}$ with equal deficiency indices and let T be a linear relation in $\mathfrak{H}$ such that $S \subset T \subset \operatorname{clos} T = S^*$. Then the triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\mathcal{H}$ is a Hilbert space and $\Gamma = \{\Gamma_0, \Gamma_1\}$ is a single-valued linear mapping from T to $\mathcal{H}^2$, is said to be a *boundary triplet of bounded type*, or *generalized boundary triplet, for* S^*, *if:*

(B1) *Green's identity* (1.1) *holds for all* $\widehat{f} = \{f, f'\}$, $\widehat{g} = \{g, g'\} \in T$;

(B2) $\operatorname{ran} \Gamma_0 = \mathcal{H}$;

(B3) $A_0 := \ker \Gamma_0$ *is a selfadjoint relation in* $\mathfrak{H}$.

The term "boundary triplet of bounded type" is used here to indicate that the Weyl function $M(\lambda)$ corresponding to the boundary triplet in Definition **??** is bounded, as is stated in the next proposition.

Proposition

Let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *be a boundary triplet of bounded type for* S^*. *Then:*

(i) $T = A_0 \,\widehat{+}\, \widehat{\mathfrak{N}}_\lambda(T)$, *where* $\widehat{\mathfrak{N}}_\lambda(T) = \widehat{\mathfrak{N}}_\lambda(S^*) \cap T$ *is dense in* $\widehat{\mathfrak{N}}_\lambda(S^*)$ *for every* $\lambda \in \mathbb{C} \setminus \mathbb{R}$;

(ii) $\operatorname{clos} \Gamma_1(A_0) = \mathcal{H}$ *and* $\overline{\operatorname{ran}}\, \Gamma = \mathcal{H}^2$;

(iii) $\widehat{\gamma}(\cdot)$ *is a* $\mathbf{B}(\mathcal{H}, \widehat{\mathfrak{N}}_\lambda)$*-valued function,* $\gamma(\cdot)$ *is a* $\mathbf{B}(\mathcal{H}, \mathfrak{N}_\lambda)$*-valued function, and* $M(\cdot)$ *is a* $\mathbf{B}(\mathcal{H})$*-valued function; each of these functions is holomorphic on* $\mathbb{C} \setminus \mathbb{R}$.

Unitary boundary triplets of bounded type (continued)

Let $R[\mathcal{H}]$ denote the class of bounded Nevanlinna functions, characterized by the conditions

(i) $M(\cdot) : \mathbb{C} \setminus \mathbb{R} \to \mathbf{B}(\mathcal{H})$ is holomorphic;

(ii) $M(\lambda)^* = M(\bar{\lambda})$ for all $\lambda \in \rho(A_0)$;

(iii) $\text{Im } M(\lambda)\text{Im }(\lambda) \geq 0$ for all $\lambda \in \rho(A_0)$.

Denote by $R^s[\mathcal{H}]$ the class of strict Nevanlinna operator-valued functions with values in $\mathbf{B}(\mathcal{H})$, that is

$$M \in R^s[\mathcal{H}] \Leftrightarrow M \in R[\mathcal{H}] \text{ and } 0 \notin \sigma_p(\mathrm{Im}\, M(\lambda)) \text{ for all } \lambda \in \mathbb{C} \setminus \mathbb{R},$$

and denote by $R^u[\mathcal{H}]$ the class of all uniformly strict functions in $R[\mathcal{H}]$, which satisfy

$$0 \in \rho(\text{Im } M(\lambda)) \quad \text{for all } \lambda \in \rho(A_0).$$

The next proposition shows that the class $R^s[\mathcal{H}]$ of bounded strict Nevanlinna functions in fact characterizes boundary triplets of bounded type; see [DHMS06] for further details.

Proposition

The Weyl function $M(\cdot)$ corresponding to a boundary triplet of bounded type $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ belongs to the class $R^s[\mathcal{H}]$. Conversely, every $R^s[\mathcal{H}]$-function is the Weyl function of some boundary triplet of bounded type $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$.

Proposition

The Weyl function $M(\cdot)$ corresponding to an ordinary boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ belongs to the class $R^u[\mathcal{H}]$. Conversely, every $R^u[\mathcal{H}]$-function is the Weyl function of an ordinary boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$.

Isometric relations in Krein spaces

A linear relation T from $(\mathfrak{H}_1, J_1)$ to $(\mathfrak{H}_2, J_2)$ is said to be *isometric* if

$$T^{-1} \subset T^{[*]}, \tag{1.5}$$

or equivalently,

$$(J_2 f', g')_{\mathfrak{H}_2} = (J_1 f, g)_{\mathfrak{H}_1} \text{ for all } \{f, f'\}, \{g, g'\} \in T. \tag{1.6}$$

The closure of an isometric relation is automatically isometric.

Lemma

Let T be an isometric relation from $(\mathfrak{H}_1, J_1)$ to $(\mathfrak{H}_2, J_2)$. Then

$$\mathrm{dom}\, T \subset \mathrm{ran}\, T^{[*]}, \quad \mathrm{ran}\, T \subset \mathrm{dom}\, T^{[*]},$$

and

$$\ker\, T \subset \ker\, (\mathrm{clos}\, T) \subset (\mathrm{dom}\, T)^{[\perp]}, \quad \mathrm{mul}\, T \subset \mathrm{mul}\, (\mathrm{clos}\, T) \subset (\mathrm{ran}\, T)^{[\perp]}.$$

An isometric relation with dense range is necessarily single-valued and so is its closure; i.e., closable. The next result is now easy to establish, cf. [DHMS2012]; also proved by Sorjonen 1980.

Proposition

Let T be a linear relation from $(\mathfrak{H}_1, J_1)$ to $(\mathfrak{H}_2, J_2)$. Then the following statements are equivalent:

(i) $T^{-1} = T^{[*]}$;

(ii) $T^{-1} \subset T^{[*]}$, $\ker\, T = (\mathrm{dom}\, T)^{[\perp]}$, *and* $\mathrm{ran}\, T = \mathrm{dom}\, T^{[*]}$;

(iii) $T^{-1} \subset T^{[*]}$, $\mathrm{mul}\, T = (\mathrm{ran}\, T)^{[\perp]}$, *and* $\mathrm{dom}\, T = \mathrm{ran}\, T^{[*]}$.

Isometric relations in Krein spaces (continued)

Corollary

Let T be a linear relation from $(\mathfrak{H}_1, J_1)$ to $(\mathfrak{H}_2, J_2)$. Then the following statements are equivalent:

(i) $T^{-1} = T^{[*]}$ *and* $\operatorname{ran} T = \mathfrak{H}_2$;

(ii) $T^{-1} \subset T^{[*]}$, $(\operatorname{dom} T)^{[\perp]} = \ker T$, *and* $\operatorname{ran} T = \mathfrak{H}_2$;

(iii) $T^{-1} \subset T^{[*]}$, $(\ker T)^{[\perp]} = \operatorname{dom} T$, *and* $\overline{\operatorname{ran}}\, T = \mathfrak{H}_2$.

In particular, one has:

Corollary

A triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is an ordinary boundary triplet for S^* if and only if $\Gamma = \{\Gamma_0, \Gamma_1\}$ is an isometric relation with $\ker \Gamma = S$ and dense domain in S^*, such that

$$\operatorname{ran} \Gamma = \mathcal{H}^2.$$

The last corollary is a slightly weaker characterization for ordinary boundary triplets than the one obtained in [DHMS06].

Isometric boundary mappings

Let Γ be an isometric relation from the Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$. Then the abstract Green's identity

$$(f', g)_{\mathfrak{H}} - (f, g')_{\mathfrak{H}} = (h', k)_{\mathcal{H}} - (h, k')_{\mathcal{H}}, \quad \{\widehat{f}, \widehat{h}\}, \{\widehat{g}, \widehat{k}\} \in \Gamma, \tag{1.7}$$

holds, where

$$\widehat{f} = \{f, f'\}, \widehat{g} = \{g, g'\} \in \mathfrak{H}^2, \ \widehat{h} = \{h, h'\}, \widehat{k} = \{k, k'\} \in \mathcal{H}^2.$$

Since Γ is isometric one has $\ker \Gamma \subset (\mathrm{dom}\,\Gamma)^{[\perp]}$ and $\mathrm{mul}\,\Gamma \subset (\mathrm{ran}\,\Gamma)^{[\perp]}$. Note that these inclusions need not hold as equalities, if Γ is not unitary.
In this general context an isometric relation $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ can be also viewed as an *isometric boundary relation for the closure of* $T = \mathrm{dom}\,\Gamma$.

Associate with Γ the component mappings Γ_0 and Γ_1 as before. Then

$$A_0 := \ker \Gamma_0, \quad A_1 := \ker \Gamma_1$$

are contained in $\mathrm{dom}\,\Gamma$ and, in general, they are non-closed symmetric extensions of $\ker \Gamma$.

To every isometric relation $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ one can associate the *Weyl family* in a similar way as in the unitary case:

$$M(\lambda) := \Gamma \widehat{\mathfrak{N}}_\lambda(T), \quad \lambda \in \mathbb{C},$$

where $\widehat{\mathfrak{N}}_\lambda(T) = \{ \widehat{f}_\lambda : \widehat{f}_\lambda = \{f_\lambda, \lambda f_\lambda\} \in \mathrm{dom}\,\Gamma\}$.

Isometric boundary mappings (continued)

Let $\widehat{h} \in M(\lambda)$ and $\widehat{k} \in M(\mu)$ with $\lambda, \mu \in \mathbb{C}$, then there exist $\widehat{f}_\lambda, \widehat{g}_\mu \in T$, such that $\{\widetilde{h}, \widehat{f}_\lambda\}$, $\{\widetilde{k}, \widehat{g}_\mu\} \in T$. Green's identity (1.1) then gives

$$(h', k)_{\mathcal{H}} - (h, k')_{\mathcal{H}} = (\lambda - \bar{\mu})(f_\lambda, g_\mu)_{\mathfrak{H}}. \tag{1.8}$$

In particular, with $\mu = \bar{\lambda}$ (1.8) implies that

$$M(\lambda) \subset M(\bar{\lambda})^*, \quad \lambda \in \mathbb{C}.$$

With $\mu = \lambda \in \mathbb{C} \setminus \mathbb{R}$ (1.8) implies that, for instance, $\ker\,(\Gamma_0 \restriction \widehat{\mathfrak{N}}_\lambda(T)) = \{0\}$. Therefore,

$$\widehat{\gamma}(\lambda) = (\Gamma_0 \restriction \widehat{\mathfrak{N}}_\lambda(T))^{-1}$$

is a single-valued mapping from $\operatorname{dom} M(\lambda)$ onto $\widehat{\mathfrak{N}}_\lambda(T)$ and, thus, as in the unitary case one can define the γ-*field* as the first component of the mapping $\widehat{\gamma}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Furthermore, (1.8) shows that $M(\lambda)$ is dissipative (accumulative) for $\lambda \in \mathbb{C}_+$ (for $\lambda \in \mathbb{C}_-$, respectively). However, observe that by definition $M(\lambda) \subset \operatorname{ran}\Gamma$, while in general $M(\lambda)^* \not\subset \operatorname{ran}\Gamma$.
The next result gives an analytic criterion for an isometric relation $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ to be a boundary relation for S^* which is based on the properties of the Weyl function M.

Theorem

The linear relation $\Gamma : \mathfrak{H}^2 \mapsto \mathcal{H}^2$ is a unitary boundary relation for S^* if and only if the following conditions hold:

(i) $\operatorname{dom}\Gamma$ *is dense in* S^**;*

(ii) Γ *is closed and isometric from* $(\mathfrak{H}^2, J_{\mathfrak{H}})$ *to* $(\mathcal{H}^2, J_{\mathcal{H}})$*;*

(iii) $\operatorname{ran}(M(\lambda) + \lambda)$ *is dense in* $\mathcal{H}$ *for some (and, hence, for all)* $\lambda \in \mathbb{C}_+$ *and for some (and, hence, for all)* $\lambda \in \mathbb{C}_-$*.*

Quasi-boundary triplets

The following definition can be seen as a modification of the notion of a *generalized boundary triplet* introduced by V. Derkach and M. Malamud in 1995.

Definition (Behrndt and Langer (2007))

Let S be a closed symmetric relation in a Hilbert space $\mathfrak{H}$ with equal deficiency indices. Then $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is said to be a *quasi-boundary triplet* for S^* if Γ_0 and Γ_1 are linear mappings defined on a dense subspace $T = \mathrm{dom}\,\Gamma$ of S^* with values in $\mathcal{H}$ such that

(Q1) Green's identity (1.7) holds for all $\widehat{f} = \{f, f'\}$, $\widehat{g} = \{g, g'\} \in T$;

(Q2) the range of $\Gamma := \{\Gamma_0, \Gamma_1\}$ is dense in $\mathcal{H}^2$;

(Q3) $A_0 := \ker \Gamma_0$ is a selfadjoint linear relation in $\mathfrak{H}$.

Thus Γ is isometric with dense range by (Q1) and (Q2). The condition (Q3) implies that $A_0 \subset T = \mathrm{dom}\,\Gamma$, and this yields the identity

$$S = \ker \Gamma = T^*.$$

An application of Corollary 14 shows that for a quasi-boundary triplet the following statements are equivalent:

(i) $\mathrm{dom}\,\Gamma = S^*$;

(ii) $\mathrm{ran}\,\Gamma = \mathcal{H}^2$;

(iii) Γ is a bounded unitary operator.

Quasi-boundary triplets (continued)

The following result gives a complete description for the class of quasi-boundary triplets, describes their closures, and expresses their connection to boundary triplets of bounded type via simple (isometric) triangular transformations on the boundary space $\mathcal{H} \times \mathcal{H}$.

Theorem

Let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet of bounded type for S^* and let E be a symmetric densely defined operator in $\mathcal{H}$. Then the transform

$$\begin{pmatrix} \widetilde{\Gamma}_0 \\ \widetilde{\Gamma}_1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ E & I \end{pmatrix} \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} \tag{1.9}$$

is a quasi-boundary triplet for S^*. Furthermore, $\widetilde{\Gamma} := \{\widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ in (1.9) is closed if and only if E is a closed symmetric operator in $\mathcal{H}$, in particular, the closure of $\widetilde{\Gamma}$ is given by (1.9) with E replaced by its closure E^{**}.

Conversely, if $\widetilde{\Gamma}$ is a quasi-boundary triplet for S^* then there exists a boundary triplet of bounded type $\Gamma = \{\Gamma_0, \Gamma_1\}$ and a densely defined symmetric operator E in $\mathcal{H}$ such that $\widetilde{\Gamma}$ is given by (1.9).

Corollary

The class of quasi-boundary triplets coincides with the class of isometric boundary triplets whose Weyl function is of the form

$$\widetilde{M}(\lambda) = E + M(\lambda), \tag{1.10}$$

with E a symmetric densely defined operator in $\mathcal{H}$ and $M \in R^s[\mathcal{H}]$.

Calkin's approach

The concept of boundary relation was introduced in 2006 by V.Derkach, S.H, M. Malamud, and H. de Snoo. Only recently we have found out that in the case that the symmetric operator is densely defined and the boundary mapping is a (single-valued) operator, the concept of boundary relation is equivalent to Calkin's notion of *reduction operator* published in 1939. In the more general context of symmetric relations his definition can be restated in the following form.

Definition

Let S be a closed symmetric relation in a Hilbert space $\mathfrak{H}$ and let $\mathfrak{M}$ be a Hilbert space. A linear operator $\Gamma : S^* \to \mathfrak{M}$ is called a *reduction operator* for S^* if

(R1) $\mathrm{dom}\,\Gamma$ is dense in S^*;

(R2) Γ is closed;

(R3) there is a signature operator J in $\mathfrak{M}$ ($J = J^* = J^{-1}$), such that

$$(\mathfrak{H} \oplus \mathfrak{H} \oplus \mathfrak{M}) \ominus \Gamma = \{\{f' \oplus -f \oplus iJ\Gamma\widehat{f}\,\} : \widehat{f} := \{f, f'\} \in \mathrm{dom}\,\Gamma\}.$$

The lefthand side in the defining identity in (R3) denotes the orthogonal complement of the graph of Γ in $\mathfrak{H} \times \mathfrak{H} \times \mathfrak{M}$. This identity gives rise to the abstract form of Green's (or Lagrange's) identity:

$$(f', g)_{\mathfrak{H}} - (f, g')_{\mathfrak{H}} = i(J\Gamma\widehat{f}, \Gamma\widehat{g}\,)_{\mathfrak{M}}, \quad \widehat{f}, \widehat{g} \in \mathrm{dom}\,\Gamma, \tag{1.11}$$

since any element in Γ has the form $\{f, f', \Gamma\widehat{f}\}$, $\widehat{f} \in \mathrm{dom}\,\Gamma \subset S^*$.

Calkin's approach (continued)

The Hilbert space $\mathfrak{M}$ together with its signature operator J gives rise to a Kreĭn space $(\mathfrak{M}, J)$ with canonical symmetry J, whereas the product space $\mathfrak{H}^2 = \mathfrak{H} \times \mathfrak{H}$ with $J_{\mathfrak{H}}$ as above gives rise to a Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ with canonical symmetry $J_{\mathfrak{H}}$. With these canonical symmetries Green's identity (1.11) can be rewritten as

$$(J_{\mathfrak{H}}\widehat{f}, \widehat{g}) = (J\Gamma\widehat{f}, \Gamma\widehat{g}), \quad \widehat{f}, \widehat{g} \in \operatorname{dom}\Gamma,$$

in other words, the operator Γ from $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to $(\mathfrak{M}, J)$ is isometric; cf. (1.6). In fact, it follows from (R2) and (R3) that the operator Γ is unitary. In particular, $\operatorname{ran}\Gamma$ is dense in $\mathfrak{M}$. Using Proposition 6 one obtains $\ker\Gamma = (\operatorname{dom}\Gamma)^{[\perp]} = (S^*)^{[\perp]} = (S^{[\perp]})^{[\perp]}$. Hence, $\ker\Gamma = S$. For the next result, use Corollary 7.

Corollary (Calkin 1939)

Let Γ be a reduction operator for S^*. Then the following statements are equivalent:

(i) *Γ is bounded unitary operator;*

(ii) $\operatorname{dom}\Gamma = S^*$;

(iii) $\operatorname{ran}\Gamma = \mathfrak{M}$.

Let S be a closed symmetric relation in a Hilbert space $\mathfrak{H}$ and let Γ be a reduction operator for S^*. If A is a proper extension of S, such that $A \subset \operatorname{dom}\Gamma$, then $\Theta := \Gamma(A)$ is a subspace of $\mathfrak{M}$ such that $\Theta \subset \operatorname{ran}\Gamma$. Conversely, if Θ is a subspace of $\mathfrak{M}$ such that $\Theta \subset \operatorname{ran}\Gamma$, then the formula

$$A_\Theta := \Gamma^{-1}\Theta \tag{1.12}$$

determines a proper extension of A, such that $A_\Theta \subset \operatorname{dom}\Gamma$. The correspondence in (1.12) simplifies under the conditions of Corollary 21.

Calkin's approach (continued)

Theorem (Calkin 1939)

Assume that S^* has a bounded reduction operator Γ. Then the formula (1.12) *establishes a one-to-one correspondence between all linear subspaces Θ of $\mathfrak{M}$ and all proper extensions A_Θ of A. Moreover,*

(i) *A_Θ is closed $\Leftrightarrow$ Θ is closed;*

(ii) *A_Θ is symmetric $\Leftrightarrow$ Θ is neutral in $(\mathfrak{M}, J)$;*

(iii) *A_Θ is maximal symmetric $\Leftrightarrow$ Θ is maximal neutral in $(\mathfrak{M}, J)$;*

(iv) *A_Θ is self-adjoint $\Leftrightarrow$ Θ is hyper-maximal neutral in $(\mathfrak{M}, J)$.*

In general reduction operators need not be bounded. If the reduction operator Γ is unbounded then there are still maximal symmetric (not necessarily selfadjoint) extensions A of S, such that $A \subset \operatorname{dom} \Gamma$ and, hence, $A = A_\Theta$ for some $\Theta \subset \mathfrak{M}$; see Theorem 4.3 in Calkin 1939. However, there are also maximal symmetric extensions $\widetilde{A}$ of S, such that $\widetilde{A} \cap \operatorname{dom} \Gamma = S$; see Theorem 4.6 in Calkin 1939.

For the connection of the present paper with Calkin's work one assumes that in Definition 20 $\mathfrak{M} = \mathcal{H} \times \mathcal{H}$ with a Hilbert space $\mathcal{H}$ so that $(\mathfrak{M}, J_\mathcal{H})$ is a Kreĭn space with the canonical symmetry $J_\mathcal{H}$ given by

$$J_\mathcal{H} = \begin{pmatrix} 0 & -iI_\mathcal{H} \\ iI_\mathcal{H} & 0 \end{pmatrix}.$$

For a detailed discussion on Calkin's main results on unbounded reduction operators we refer to the paper by S.H. & H. Wietsma (2012).

- Behrndt, J. and Langer, M. 2007 Boundary value problems for elliptic partial differential operators on bounded domains. *J. Functional Analysis*, **243**, 536–565.
- Calkin, J.W. 1939. Abstract symmetric boundary conditions. *Trans. Amer. Math. Soc.*, **45**, 369–442
- Derkach, V.A., Hassi, S., Malamud, M.M., and de Snoo, H.S.V. 2006. Boundary relations and Weyl families. *Trans. Amer. Math. Soc.*, **358**, 5351–5400.
- Derkach, V.A., Hassi, S., Malamud, M.M., and de Snoo, H.S.V. 2009. Boundary relations and generalized resolvents of symmetric operators. *Russ. J. Math. Phys.*, **16**, 17–60.
- V.A. Derkach, S. Hassi, M.M. Malamud, and H.S.V. de Snoo, "Boundary triplets and Weyl functions. Recent developments", London Mathematical Society Lecture Notes, 404, (2012), 161–220.
- Derkach, V.A., and Malamud, M.M. 1991. Generalized resolvents and the boundary value problems for hermitian operators with gaps. *J. Funct. Anal.*, **95**, 1–95.
- Derkach, V.A., and Malamud, M.M. 1995. The extension theory of hermitian operators and the moment problem. *J. Math. Sciences*, **73**, 141–242.
- S. Hassi and R. Wietsma, "On Calkin's abstract symmetric boundary conditions", London Mathematical Society Lecture Notes, 404, (2012), 3–34.