Quadratic operators on AM-spaces

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Definition

Let $(X, +), (Y, +)$ be Abelian groups. A map $Q\colon X \to Y$ is termed *quadratic* if it satisfies the following functional equation:

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y), \quad x, y \in X. \qquad (1)$$

Theorem (Aczél)

Let $(X, +), (Y, +)$ be Abelian groups and let assume that $(Y, +)$ is uniquely divisible by 2. A map $Q\colon X \to Y$ is quadratic if and only if there exists a bi-additive and symmetric mapping $B\colon X \times X \to Y$ such that

$$Q(x) = B(x, x), \quad x \in X.$$

Moreover, B is uniquely determined via the following formula:

$$B(x, y) = \frac{1}{4}[Q(x + y) - Q(x - y)], \quad x, y \in X. \tag{2}$$

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Assume that X is a linear space over a field K and $T\colon X \to K$ is an additive function. For an arbitrary constant $c \in K$ the map $Q\colon X \to K$ given by:

$$Q(x) = cT(x)^2, \quad x \in X \tag{3}$$

is an example of a quadratic mapping.

Each mapping $Q\colon X \to K$ given by (3) with some additive $T\colon X \to K$ and some constant $c \in K$ satisfies:

$$[Q(x+y) - Q(x-y)]^2 = 16Q(x)Q(y), \quad x, y \in X. \tag{4}$$

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Theorem (Kurepa & Ebanks)

Let $(X, +)$ *be a group,* K *a field with characteristic different from 2 and* $Q\colon X \to K$ *a quadratic map. Then,* Q *satisfies equation* (4) *if and only if there exist an additive map* $T\colon X \to K$ *and a nonzero constant* $c \in K$ *such that the formula* (3) *holds.*

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Assume that $Q\colon \mathbb{R} \to \mathbb{R}$ is an arbitrary mapping. Then Q is a quadratic-multiplicative function if and only if there exists an additive-multiplicative function $T\colon \mathbb{C} \to \mathbb{C}$ such that Q is of the form

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If K is a field then we denote by $\overline{K}$ the algebraic closure of K and if $\zeta \in \overline{K}$, then $K(\zeta)$ stands for the smallest field such that $K \subseteq K \cup \{\zeta\} \subseteq \overline{K}$.

Theorem (Gajda)

Assume that X is a commutative unitary ring, K is a field with characteristic different from 2 and $Q\colon X \to K$ is an arbitrary mapping. Then Q is a quadratic-multiplicative function if and only if there exist an element $\zeta \in \overline{K}$ such that $\zeta^2 \in K$ and additive-multiplicative mappings $u\colon X \to K(\zeta)$ and $v\colon X \to K(\zeta)$ such that

$$u(x) + v(x) \in K, \qquad u(x) - v(x) \in \zeta K, \; x \in X$$

and Q is of the form

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A Banach lattice X is called *AM-space* if

$$\|x \vee y\| = \max\{\|x\|, \|y\|\}, \quad x, y \in X^+ \text{ with } x \wedge y = 0.$$

The Kakutani-Bohnenblust-Krein theorem says that every *AM*-space with a unit is lattice-isometric to the space $C(\Omega)$ with some compact Hausdorff space Ω and moreover every *AM*-space is lattice-isometric to some closed vector sublattice of $C(\Omega)$.

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Theorem

Assume that Ω_1, Ω_2 *are compact Hausdorff spaces. Then* $Q\colon C(\Omega_1) \to C(\Omega_2)$ *is a quadratic and multiplicative operator if and only if there exist a clopen subset* $B \subseteq \Omega_2$ *and mappings* $\tau, \sigma\colon \Omega_2 \to \Omega_1$ *which are continuous on* B *such that:*

$$Q(x)(t) = \chi_B(t)x(\tau(t))x(\sigma(t)), \quad x \in C(\Omega_1),\ t \in \Omega_2. \tag{6}$$

Theorem

Assume that Ω_1, Ω_2 are compact Hausdorff spaces and Ω_2 is metrizable. and $Q\colon C(\Omega_1) \to C(\Omega_2)$ is an arbitrary map. Then Q is a nonnegative and continuous quadratic operator which satisfies equality (4) jointly with the following auxiliary condition:

$$Q(x+y) = Q(x-y) \text{ for all } x, y \in \mathbb{C}(\Omega_1) \text{ such that } x \wedge y = 0 \quad (7)$$

if and only if there exist a mapping $\tau\colon \Omega_2 \to \Omega_1$ and a nonnegative function $w \in C(\Omega_2)$ such that τ is continuous on the set $\{t \in \Omega_2 : w(t) > 0\}$ and:

$$Q(x)(t) = w(t)[x(\tau(t))]^2, \quad x \in C(\Omega_1),\ t \in \Omega_2. \qquad (8)$$

Theorem

Assume that X is a vector lattice, Y is an Abelian group, $Q\colon X \to Y$ is a quadratic map and $B\colon X \times X \to Y$ is the corresponding bi-additive and symmetric mapping. Then the following conditions are equivalent:

(i) $B(x, y) = 0$ *for all* $x, y \in X$ *such that* $x \wedge y = 0$;

(ii) $B(x^+, x^-) = 0$ *for all* $x \in X$;

(iii) $Q(x^+) + Q(x^-) = Q(|x|)$ *for all* $x \in X$;

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Theorem

Assume that X, Y are vector lattices, $Q\colon X \to Y$ is a quadratic map and $B\colon X \times X \to Y$ is the corresponding bi-additive and symmetric mapping. Then the following conditions are equivalent:

- (i) $2B(x,y) = Q(x-y)^-$ *for all* $x, y \in X$ *such that* $x \wedge y = 0$;
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(v) $|Q(x)| = Q(|x|)$ *for all* $x \in X$;

(vi) $|Q(x - y)| = Q(x + y)$ *for all* $x, y \in X$ *such that* $x \wedge y = 0$.

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