Norm equalities in pre-Hilbert C^*-modules

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Pre-Hilbert C^*-Modules

Definition

A pre-Hilbert module over the C^*-algebra A (or a pre-Hilbert A-module) is a complex vector space E which has a compatible structure of a right A-module and it is endowed with an A-valued inner product, i.e., with a map

$$E \times E \ni (x, y) \mapsto \langle x, y \rangle \in A$$

that satisfies the conditions:

1. $\langle x, x \rangle \geq 0, \; x \in E$; $\langle x, x \rangle = 0$ iff $x = 0$;
2. $\langle x, y \rangle^* = \langle y, x \rangle, \; x, y \in E$;
3. $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle, \; \alpha, \beta \in \mathbb{C}, \; x, y, z \in E$;
4. $\langle x, ya \rangle = \langle x, y \rangle a, \; a \in A, \; x, y \in E$.

Cauchy-Schwarz Inequalities

Definition

E is a Hilbert *A*-module if the induced norm

$$E \ni x \mapsto \|x\| := \|\langle x, x\rangle\|_A^{1/2} \in \mathbb{R}_+$$

is complete.

Cauchy-Schwarz Inequality

The following inequality holds true in general pre-Hilbert C^*-modules:

$$\langle x, y\rangle^* \langle x, y\rangle \leq \|x\|^2 \langle y, y\rangle, \quad x, y \in E.$$

By passing to norms we also get:

$$\|\langle x, y\rangle\|_A \leq \|x\| \|y\|, \quad x, y \in E.$$

Basic Examples

1. Complex Hilbert spaces are, according to the definition above, Hilbert $\mathbb{C}$-modules.
2. Any C^*-algebra A can be regarded as a Hilbert module over itself with respect to the inner product

$$A \times A \ni (a, b) \mapsto a^* b \in A.$$

3. Given two Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, the space $\mathcal{B}(\mathscr{H}_1, \mathscr{H}_2)$ is a Hilbert $\mathcal{B}(\mathscr{H}_1)$-module when endowed with the inner product

$$\mathcal{B}(\mathscr{H}_1, \mathscr{H}_2) \times \mathcal{B}(\mathscr{H}_1, \mathscr{H}_2) \ni (S, T) \mapsto S^* T \in \mathcal{B}(\mathscr{H}_1).$$

The “Rank One” Operator

Let E and F be two pre-Hilbert modules over A.

Adjointable Maps

A map $T : E \to F$ is said to be adjointable if there exists $T^* : F \to E$ (called the adjoint of T) such that

$$\langle Tx, y\rangle_F = \langle x, T^*y\rangle_E, \quad x \in E,\ y \in F.$$

Rank One Operators

For given $x \in E$ and $y \in F$ the “rank one” operator is defined as

$$F \ni z \mapsto \theta_{x,y}(z) := x\langle y, z\rangle_F \in E$$

Its basic properties are:

- $\theta_{x,y}$ is adjointable ($\theta^*_{x,y} = \theta_{y,x}$);
- $\|\theta_{x,y}\| \leq \|x\|\|y\|,\ x \in E, y \in F$; $\|\theta_{x,x}\| = \|x\|^2$ if $x \in E = F$.

The Triangle Equality for “Rank One” Operators

Proposition

Let E be a pre-Hilbert module over the C^*-algebra A and $x, y \in E$. Then

$$\|\theta_{x,x} + \theta_{y,y}\| = \left\| \begin{pmatrix} \langle x, x\rangle & \langle x, y\rangle \\ \langle y, x\rangle & \langle y, y\rangle \end{pmatrix} \right\|_{\mathscr{M}_2(A)} .$$

Theorem

Let E be a pre-Hilbert A-module and x, y be two nonnull elements of E. The following conditions are equivalent:

(i) $\|\theta_{x,x} + \theta_{y,y}\| = \|\theta_{x,x}\| + \|\theta_{y,y}\|$;

(ii) $\|\langle x, y\rangle\|_A = \|x\|\|y\|$.

The Triangle Equality

Theorem

Let E be a pre-Hilbert module over the C^*-algebra A and $x, y \in E$. The following conditions are equivalent:

(*i*) $\|x + y\| = \|x\| + \|y\|$;

(*ii*) $\|x\|\|y\| \in \sigma(\Re\langle x, y\rangle)$.

Triangle vs. Cauchy-Schwarz Equalities

Corollary

Let E be a pre-Hilbert module over the C^*-algebra A and $x, y \in E$ such that $\Re\langle x, y\rangle \geq 0$. The following conditions are equivalent:

(*i*) $\|x + y\| = \|x\| + \|y\|$;

(*ii*) $\|\Re\langle x, y\rangle\| = \|x\|\|y\|$.

Corollary

Let E be a pre-Hilbert module over the C^*-algebra A and $x, y \in E$ such that $r(\langle x, y\rangle) \in \sigma(\langle x, y\rangle)$ (in particular, $\sigma(\langle x, y\rangle) \subseteq [0, \infty)$). The following conditions are equivalent:

(*i*) $\|x + y\| = \|x\| + \|y\|$;

(*ii*) $\|\Re\langle x, y\rangle\| = \|x\|\|y\|$.

Positive Elements in a C^*-Algebra

Corollary (F. Kittaneh, 2002)

Let a, b be positive elements of a C^*-algebra A. The following conditions are equivalent:

(i) $\|a + b\| = \|a\| + \|b\|$;

(ii) $\|ab\| = \|a\|\|b\|$.

Pythagoras' equality

Theorem

Let $x_1, \ldots, x_n$ be elements in a pre-Hilbert C^*-module such that $\Re\langle x_i, x_j\rangle = 0$ for $i, j \in \{1, \ldots, n\},\ i \neq j$. The following conditions are equivalent:

(*i*) $\left\|\sum_{j=1}^{n} x_j\right\|^2 = \sum_{j=1}^{n} \|x_j\|^2;$

(*ii*) $\left\|\prod_{j=1}^{n} \langle x_j, x_j\rangle\right\|_A = \prod_{j=1}^{n} \|x_j\|^2.$

Idempotent Inner Products - 1

Theorem

Let x, y be two elements in a pre-Hilbert module over the unital C^*-algebra A such that, for some $\alpha \in \mathbb{C}$, $\langle x, y\rangle - \alpha \notin \{0, 1\}$ is an idempotent. The following conditions are equivalent:

(*i*) $\|x + y\| = \|x\| + \|y\|$;

(*ii*) $\alpha = \|x\|\|y\| - 1 \geq -\frac{1}{2}$.

Proposition

Let x, y be two elements in a pre-Hilbert module over the unital C^*-algebra A such that $\langle x, y\rangle \in \mathbb{C} \cdot 1$. The following conditions are equivalent:

(*i*) $\|x + y\| = \|x\| + \|y\|$;

(*ii*) $\langle x, y\rangle = \|x\|\|y\|$.

Idempotent Inner Products - 2

Corollary (L. Arambašić - R. Rajić, 2006)

Let x, y be two elements in a pre-Hilbert C^*-module such that $\langle x, y \rangle$ is a nonnull idempotent. The following conditions are equivalent:

(i) $\|x + y\| = \|x\| + \|y\|$

(ii) $\|x\|\|y\| = 1$.

Idempotent Inner Products - 3

Corollary

Let $a \notin \{0, 1\}$ be an idempotent in a unital C^*-algebra A. The following conditions are equivalent:

(*i*) *a is a selfadjoint projection*;

(*ii*) *There exists (for every) $\alpha \geq -\frac{1}{2}$ such that (it holds)*

$$\|a + \alpha\| = 1 + \alpha;$$

(*iii*) *There exists (for every) $\alpha \geq -\frac{1}{2}$ such that (it holds)*

$$\|a + \alpha + 1\| = \|a + \alpha\| + 1.$$

Isometric Inner Products - 1

Theorem

Let x, y be two elements in a pre-Hilbert module over a unital C^*-algebra such that $\langle x, y \rangle$ is a nonunitary isometry. The following conditions are equivalent:

(i) $\|x + y\| = \|x\| + \|y\|$

(ii) $\|x\|\|y\| = 1$.

Corollary

Let A be a unital C^*-algebra, $u \in A$ unitary and $v \in A$ a nonunitary isometry. Then

$$\|u - v\| = 2.$$

Isometric Inner Products - 2

Corollary

Let $\mathscr{H}$ be a Hilbert space and $U, S \in \mathcal{B}(\mathscr{H})$ such that U is unitary and S a shift. Then

$$\|U - S\| = 2.$$

Corollary

Let A be a unital C^*-algebra, $v \in A$ an isometry and $w \in A$ a co-isometry such that at least one of them is not unitary. Then

$$\|u - w\| = 2.$$

Thank you for your attention!

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