Two classical boundedness results for pseudodifferential operators: the anisotropic case

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Joint work with Marcin Bownik, University of Oregon

Outline of the talk

- Some examples
- Anisotropic symbols from diagonal matrices
- Anisotropic symbols from general matrices
- Boundedness results

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Example 1

$$\sigma(\xi_1, \xi_2) = \frac{\xi_1^6 + \xi_2^6}{\xi_1^6 + \xi_2^2 + \xi_2^6}, \quad (\xi_1, \xi_2) \neq (0, 0)$$

Question

What kind of estimates does σ satisfy?

Clearly,

$$|\sigma(\xi_1, \xi_2)| = \sigma(\xi_1, \xi_2) \leq 1$$

Now, the derivatives; for example:

$$|\partial_{\xi_1}\sigma(\xi_1, \xi_2)| = \frac{6|\xi_1|^5|\xi_2|^2}{(\xi_1^6 + \xi_2^2 + \xi_2^6)^2}$$

Consider two cases:

1. $|\xi_1|^3 \geq |\xi_2|$
2. $|\xi_1|^3 \leq |\xi_2|$

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Example 1: derivative estimates

Case 1: $|\xi_1|^3 \geq |\xi_2|$

$$\frac{|\xi_1|^5|\xi_2|^2}{(\xi_1^6+\xi_2^2+\xi_2^6)^2} \leq \frac{|\xi_1|^6|\xi_1|^{-1}|\xi_2|^2}{|\xi_1|^{12}} \leq \left(\frac{|\xi_2|}{|\xi_1|^3}\right)^2 |\xi_1|^{-1} \leq |\xi_1|^{-1}$$

Case 2: $|\xi_1|^3 \leq |\xi_2|$

$$\frac{|\xi_1|^5|\xi_2|^2}{(\xi_1^6+\xi_2^2+\xi_2^6)^2} \leq \frac{|\xi_2|^{5/3}|\xi_2|^2}{|\xi_2|^4} \leq |\xi_2|^{-1/3}$$

We have shown:

$$|\partial^1_{\xi_1}\partial^0_{\xi_2}\sigma(\xi_1,\xi_2)| \lesssim \left(\max(|\xi_1|^2,|\xi_2|^{2/3})\right)^{-\frac{1}{2}\cdot 1}$$

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We have shown:

$$|\partial_{\xi_1}^1 \partial_{\xi_2}^0 \sigma(\xi_1,\xi_2)| \lesssim \left(\max(|\xi_1|^2, |\xi_2|^{2/3})\right)^{-\frac{1}{2}\cdot 1}$$

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Example 1 continued

Similar calculations hold for all derivatives. Specifically:

$$|\partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2}\sigma(\xi_1,\xi_2)| \lesssim \Big(\max(|\xi_1|^2, |\xi_2|^{2/3}) \Big)^{-\frac{1}{2}\beta_1 - \frac{3}{2}\beta_2}$$

for all multi-indices

$$\beta = (\beta_1, \beta_2), \quad |\beta_1| + |\beta_2| \le 2.$$

If we write $\|(\beta_1, \beta_2)\| = \frac{1}{2}\beta_1 + \frac{3}{2}\beta_2$,

Estimates

$$|\partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2}\sigma(\xi_1,\xi_2)| \lesssim \Big(\max(|\xi_1|^2, |\xi_2|^{2/3}) \Big)^{-\|(\beta_1,\beta_2)\|}$$

Example 2

$$\sigma((x_1, x_2), (\xi_1, \xi_2)) = \frac{\xi_1^6 + \varphi(x_2)\xi_2^6}{1 + \xi_1^6 + \xi_2^2 + \varphi(x_2)\xi_2^6},$$

where $\varphi \in C^\infty(\mathbb{R})$ is such that, for some constants $C_1, C_2, C_3 > 0$, for all $x \in \mathbb{R}$ and for all $k \in \mathbb{N}$:

$$C_1 \leq \varphi(x) \leq C_2, \ |\varphi^{(k)}(x)| \leq C_3.$$

For example, $\varphi(x) = 2 + \sin x$ satisfies these conditions.

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The estimates

Similar calculation prove the following.

Claim

Let $x = (x_1, x_2), \xi = (\xi_1, \xi_2)$. Then, we have

$$|\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2}\sigma(x,\xi)| \lesssim (1+\max(|\xi_1|^2, |\xi_2|^{2/3}))^{-\|(\beta_1,\beta_2)\|},$$

where $\|(\beta_1, \beta_2)\| = \frac{1}{2}\beta_1 + \frac{3}{2}\beta_2$. Or, more concisely

$$|\partial_x^{\alpha}\partial_\xi^{\beta}\sigma(x,\xi)| \lesssim (1+\max(|\xi_1|^2, |\xi_2|^{2/3}))^{-\|\beta\|}.$$

Anisotropic symbols from diagonal (dilation) matrices

Question

Why the name anisotropic?

Because the variables in the estimates do not have the same homogeneity. There is however a *natural scaling* present in Examples 1 and 2: cubic.
The estimates in Example 1 and 2 are (with $m = 0, \delta = 0, \gamma = 1$):

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \lesssim \Big((0 \text{ or } 1) + \rho_{A^*}(\xi)\Big)^{m+\delta\|\alpha\|-\gamma\|\beta\|},$$

where $x, \xi \in \mathbb{R}^n$, ρ_{A^*} is a specific *quasi-norm* and

$$\|\alpha\| = \frac{1}{a}\sum_{j=1}^n a_j\alpha_j, \text{where } a_j > 0, \sum_{j=1}^n a_j = na.$$

In Examples 1 and 2: $n = 2, a_1 = 1/2, a_2 = 3/2, a = 1.$

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Connection with diagonal matrices

Let $\lambda > 1$, $a_1, \ldots, a_n > 0$, $a_1 + \cdots + a_n = na$:

$$A = D(\lambda; a) := \begin{pmatrix} \lambda^{a_1} & 0 & \ldots & 0 \\ 0 & \lambda^{a_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda^{a_n} \end{pmatrix}$$

The eigenvalues are obviously *all strictly larger than 1*. Thus, this matrix is an example of a (diagonal) *dilation* or *expansive* matrix.

The associated quasi-norm:

$$\rho_A(x_1, \ldots, x_n) = \max_{1 \le j \le n} |x_j|^{a/a_j}$$

Alternately: $\rho_A(x_1, \ldots, x_n) = (|x_1|^{2/a_1} + \cdots + |x_n|^{2/a_n})^{a/2}$.
We have the homogeneity condition

$$\rho_A(Ax) = \rho_A(\lambda^{a_1} x_1, \ldots, \lambda^{a_n} x_n) = \lambda^a \rho_A(x) = |\det A|^{1/n} \rho_A(x).$$

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Comparison to the classical Hörmander classes

$\sigma \in S^m_{a;\gamma,\delta}$ (*inhomogeneous anisotropic*) means

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \lesssim \Big(1 + \rho_{A^*}(\xi)\Big)^{m+\delta\|\alpha\|-\gamma\|\beta\|},$$

where, as before, $A = D(\lambda; a)$

• Garello ('98); Yamazaki ('86); Leopold ('86); Lascar ('77); Hunt-Pirou ('69): boundedness on classical function spaces, symbolic calculus; some applications to PDEs

• $\sigma \in \dot{S}^m_{a;\gamma,\delta}$ (*homogeneous anisotropic*): same condition w/out + 1

These classes of symbols resemble the (*isotropic*) Hörmander ones. For example, $\sigma \in S^m_{\gamma,\delta}$ if

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The set-up

A (square) matrix A is called a *dilation* if all its *eigenvalues* λ *satisfy* $|\lambda| > 1$. We also denote by $|\cdot|$ the standard norm of $\mathbb{R}^n$.

Let P be some non-degenerate $n \times n$ matrix. There exists an *ellipsoid*

$$\Delta = \{x \in \mathbb{R}^n : |Px| < 1\}$$

such that

1. $\Delta \subset r\Delta \subset A\Delta$ for some $r > 1$
2. $|\Delta| = 1$

Define a *family of dilated balls around the origin*:

$$B_k = A^k \Delta, k \in \mathbb{Z}$$

that satisfy, with $b = |\det A|$,

$$B_k \subset rB_k \subset B_{k+1} \text{ and } |B_k| = b^k.$$

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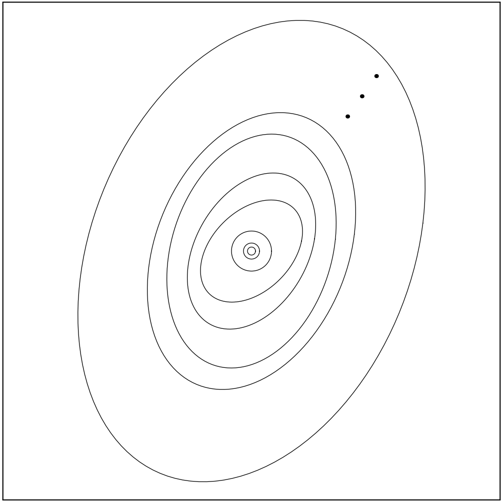

Quasi-norm associated to a dilation

Definition (Lemarié-Rieusset; Bowník)

A *homogeneous quasi-norm* associated with a dilation A is a measurable mapping $\rho : \mathbb{R}^n \to [0, \infty)$ such that

1. $\rho(x) = 0 \Leftrightarrow x = 0$
2. $\rho(Ax) = b\rho(x), x \in \mathbb{R}^n$ (recall $b := |\det A|$)
3. For some $C > 0$, $\rho(x + y) \leq C(\rho(x) + \rho(y)), x, y \in \mathbb{R}^n$.

Lemma

Any two homogeneous quasi-norms ρ_1, ρ_2 associated with a dilation A are equivalent, i.e.,

$$\exists K > 1 : \ \rho_1(x)/K \leq \rho_2(x) \leq K\rho_1(x), \ \ x \in \mathbb{R}^n.$$

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The "canonical" quasi-norm

Definition

The *step homogeneous quasi-norm* induced by A is defined by

$$\rho_A(x) = b^j \text{ for } x \in B_{j+1} \setminus B_j, \text{ and } \rho_A(0) = 0.$$

Example

Let $A = 2I_n$. Then $\rho(x) = |x|^n$ is a quasi-norm associated with A. In fact, a quasi-norm induced by a dilation A is equivalent to $|\cdot|^n$ if and only if A is diagonalizable over $\mathbb{C}$ with all eigenvalues equal in the absolute value (classification of dilations inducing the usual isotropic homogeneous structure on $\mathbb{R}^n$).

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Definition of the anisotropic class $S^m_{\gamma,\delta}(A)$

Definition

$\sigma \in S^m_{\gamma,\delta}(A)$ means

$$|\partial_x^\alpha \partial_\xi^\beta [\sigma(A^{-k_1}\cdot, (A^*)^{k_2}\cdot)](A^{k_1}x, (A^*)^{-k_2}\xi)| \lesssim (1+\rho_{A^*}(\xi))^m,$$

for all multi-indices α, β and $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$. Here,

$$k_1 = \lfloor k\delta \rfloor, \qquad k_2 = \lfloor k\gamma \rfloor,$$

where $k \in \mathbb{N}_0$ is such that $1 + \rho_{A^*}(\xi) \sim b^k$.

The derivatives above should be interpreted as

$$(\partial_x^\alpha \partial_\xi^\beta \tilde{\sigma})(A^{k_1}x, (A^*)^{-k_2}\xi),$$

where

$$\tilde{\sigma}(x,\xi) = \sigma(A^{-k_1}x, (A^*)^{k_2}\xi).$$

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$$\tilde{\sigma}(x,\xi) = \sigma(A^{-k_1}x, (A^*)^{k_2}\xi).$$

$\sim$ has the following interpretation: we pick k to be the unique non-negative integer such that the frequency variable ξ belongs to the annulus $B^*_{k+1} \setminus B^*_k$ if $k > 0$, or the ball B^*_1 if $k = 0$.

Note: we require estimates on the derivatives of a symbol σ that hold uniformly after appropriate rescaling depending on the location of the frequency variable ξ. Small detail...technical difficulties in proofs.

Similar definition for $\dot{S}^m_{\gamma,\delta}(A)$. Here, $k \in \mathbb{Z}$.

Remark

Our definition recovers

1. $S^m_{\gamma,\delta}$ *when* $A = 2I_n$
2. $S^m_{a;\gamma,\delta}$ *when* $A = D(2, a)$.

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Similar definition for $\dot{S}^m_{\gamma,\delta}(A)$. Here, $k \in \mathbb{Z}$.

Remark

Our definition recovers

1. $S^m_{\gamma,\delta}$ *when* $A = 2I_n$
2. $S^m_{a;\gamma,\delta}$ *when* $A = D(2, a)$.

Associated to any symbol $\sigma(x,\xi)$ we have a *pseudodifferential operator*

$$(\sigma(x,D)f)(x) = \int_{\mathbb{R}^n} \sigma(x,\xi)\widehat{f}(\xi)e^{ix\cdot\xi}\, d\xi.$$

When $\sigma(x,\xi) = \sigma(\xi)$, $\sigma(x,D) = \sigma(D)$ is a *multiplier operator.*

A priori: if $\sigma \in S^m_{\gamma,\delta}(A)$, then $\sigma(x,D) : \mathcal{S} \to \mathcal{S}$.

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Anisotropic Mihlin multipliers: $\dot{S}^0_1(A)$

$$|\partial_\xi^\beta[\sigma((A^*)^k\cdot)]((A^*)^{-k}\xi)| \le C_\beta \qquad \text{for all } \beta,$$

for all $\xi \in \mathbb{R}^n \setminus \{0\}$, and $k \in \mathbb{Z}$ such that $\rho_{A^*}(\xi) \sim b^k = |\det A|^k$.

In the isotropic case ($A = 2I_n$), these conditions take the familiar form

$$|\partial_\xi^\beta \sigma(\xi)| \le C_\beta |\xi|^{-|\beta|} \qquad \text{for all } \beta.$$

Example 1 revisited

$$\sigma(\xi_1, \xi_2) = \frac{\xi_1^6 + \xi_2^6}{\xi_1^6 + \xi_2^2 + \xi_2^6}$$

We have: $\sigma \in \dot{S}^0_1(A)$, with

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Boundedness results: $\dot{S}^0_1(A)$

Theorem

If $\sigma \in \dot{S}^0_1(A)$, then $\sigma(D)$ extends as a bounded operator

1. $\sigma(D) : L^p \to L^p, p > 1,$
2. $\sigma(D) : H^p \to H^p, 0 < p \leq 1.$

The proof follows from the following observation: $\sigma(D)$ is a Calderón-Zygmund operator with respect to the dilation A and the canonical quasi-norm ρ_A. More precisely:

Proposition

If $\sigma \in \dot{S}^0_1(A)$, then $K = \mathcal{F}^{-1}\sigma$ is a Calderón-Zygmund kernel:

$$|\partial^\alpha[K(A^k\cdot)](A^{-k}x)| \leq \frac{C_\alpha}{\rho_A(x)}$$

for some $C_\alpha > 0$, all multi-indices α, and all $x \in \mathbb{R}^n \setminus \{0\}$ such that $\rho_A(x) \sim b^k$.

Boundedness results: $\dot{S}_1^0(A)$

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Anisotropic Coifman-Meyer: $S^0_{1,0}(A)$

Example 2 revisited

$$\sigma((x_1, x_2), (\xi_1, \xi_2)) = \frac{\xi_1^6 + \varphi(x_2)\xi_2^6}{1 + \xi_1^6 + \xi_2^2 + \varphi(x_2)\xi_2^6}$$

We have: $\sigma \in S^0_{1,0}(A)$, with

$$A = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 2\sqrt{2} \end{pmatrix}.$$

Anisotropic Calderón-Vaillancourt: $S^0_{\delta,\delta}(A)$

Theorem

If $\sigma \in S^0_{\delta,\delta}(A)$ for some $0 \leq \delta < 1$, then the pseudodifferential operator $\sigma(x, D)$ extends to a bounded operator on L^2, and hence, also on L^p.

Remark

$S^{m_1}_{\gamma_1,\delta_1}(A) \subset S^{m_2}_{\gamma_2,\delta_2}(A)$ if $m_1 \leq m_2$, $\gamma_2 \leq \gamma_1$, and $\delta_1 \leq \delta_2$. Thus, the theorem also holds for the class $S^0_{\gamma,\delta}(A)$, where $0 \leq \delta \leq \gamma \leq 1$ and $\delta < 1$. The condition on the indices defining the classes of symbols is known to be sharp in the isotropic setting; Kumano-go ('70) and Hörmander ('71).

Idea of proof: almost orthogonality. Decompose $\sigma(x, D)$ in the frequency domain as a sum of operators $\sigma_j(x, D)$. Then show that the operators $\sigma_j(x, D)$ are very close to being mutually orthogonal. There is an alternate proof of the L^p boundedness of the Coifman-Meyer class using "elementary symbols".

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Thank you!

Conclusion

This work aimed at showing plausibility of a larger theory of anisotropic pseudo-differential operators.

Already the definition of anisotropic classes of symbols is complicated. Hence, it is expected that the proofs are technically involved.

Future work? For example:

Question

What happens when we consider more "exotic" symbols $S^{m}_{\gamma,\delta}(A)$, $m < 0$ (and depending on the other indices)? The isotropic case was settled by Fefferman ('73).

Some additional calculations

Arriving to the correct definition

Consider a symbol $\sigma \in \dot{S}^0_{1,0}$, x-independent (Mihlin multiplier):

$$|\partial^\alpha_\xi \sigma(\xi)| \lesssim |\xi|^{-|\alpha|}.$$

Assume $|\xi| \sim 2^k \Leftrightarrow 2^k \leq |\xi| < 2^{k+1}$ $(\xi \in B_{k+1} \setminus B_k)$. Define

$$\tilde{\sigma}(\xi) = \sigma(2^k \xi).$$

Then: $\partial^\alpha_\eta \tilde{\sigma}(\eta) = 2^{k|\alpha|}(\partial^\alpha_\eta \sigma)(2^k \eta)$. Then:

$$|(\partial^\alpha_\xi \tilde{\sigma})(2^{-k}\xi)| = 2^{k|\alpha|}|\partial^\alpha_\xi \sigma(\xi)| \lesssim 2^{k|\alpha|}|\xi|^{-|\alpha|} \lesssim 1.$$

Thus, the Mihlin condition looks like: for $\rho_{A^*}(\xi) \sim 2^k$,

$$|\partial^\alpha_\xi \sigma((A^*)^k \cdot)((A^*)^{-k}\xi)| \lesssim (\rho_{A^*}(\xi))^0.$$