p–Laplacian problems with nonlinearities interacting with the spectrum

joint works with Anna Maria Candela and Dora Salvatore

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Model problem

$1 < p < +\infty$, $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$

$$(P) \qquad \begin{cases} -\Delta_p u = g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Background material: Rabinowitz, Perera, Dinca-Jebelean-Mawhin (super $(p-1)$-polynomial growth)

- $\Omega \subset \mathbb{R}^N$ open bounded with smooth boundary $\partial\Omega$ $(N \geq 3)$
- $g : \Omega \times \mathbb{R} \to \mathbb{R}$ subcritical ($p^* = \frac{Np}{N-p}$ if $p \in]1, N[$, $p^* = +\infty$ otherwise)

$$\lim_{|t|\to+\infty} \frac{g(x,t)}{|t|^{p-2}t} \in \mathbb{R} \quad \text{uniformly with respect to } x \in \Omega$$

The existence of (non-trivial) solutions is related to the interaction with $\sigma(-\Delta_p)$

Overview

* $p = 2$ semilinear case: $W_0^{1,2}(\Omega)$, $\sigma(-\Delta)$
  - *existence results:*
    Amann-Zehnder, Landesman-Lazer, Ahmad-Lazer-Paul
  - *multiplicity results:*
    Rabinowitz, P.Bartolo-Benci-Fortunato, Chang

* $p \neq 2$ quasilinear case: $W_0^{1,p}(\Omega)$, $\sigma(-\Delta_p)$
  - *existence results:*
    Arcoya-Orsina, Drábek-Robinson, Li-Zhou, Liu-Li
  - *multiplicity results:*
    Li-Zhou, Perera-Szulkin

Setting of the problem

Let $l_\infty \in \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$g(x,t) \ = \ l_\infty |t|^{p-2} t + f(x,t) \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}$$

Problem (P) becomes

$$(P_\infty) \qquad \begin{cases} -\Delta_p u - l_\infty |u|^{p-2} u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover we assume:

- $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$
- $f(x,t) = o(|t|^{p-2} t)$ as $|t| \to +\infty$

Multiplicity results

The weak solutions of (P_∞) are the critical points of the C^1 functional

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \frac{l_\infty}{p}\int_\Omega |u|^p \,\mathrm{d}x - \int_\Omega F(x,u)\,\mathrm{d}x$$

on $W_0^{1,p}(\Omega)$, with $F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s$

Moreover we assume

- $\lim_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} = l_0 \in \mathbb{R}$
- $l_\infty \notin \sigma(-\Delta_p)$
- $f(x,\cdot)$ is odd for $x \in \Omega$

Semilinear case

Under the previous assumptions, if

- there exist two integers h, $k \geq 1$ such that

$$\min\{l_0 + l_\infty, l_\infty\} \ < \ \lambda_h \ < \ \lambda_k \ < \max\{l_0 + l_\infty, l_\infty\}$$

with $(\lambda_k)_k$ eigenvalues (here distinct) of $-\Delta$ in $W_0^{1,2}(\Omega)$ then

(P_∞) has at least $\dim(M_h \oplus \ldots \oplus M_k)$ distinct pairs of nontrivial solutions

where M_j the eigenspace corresponding to the eigenvalue λ_j of $-\Delta$ in $W_0^{1,2}(\Omega)$

Quasilinear case

Under the previous assumptions, if

- there exist two integers $h \geq k \geq 1$ such that

$$\min\{l_0 + l_\infty, l_\infty\} < \eta_h \leq \nu_k < \max\{l_0 + l_\infty, l_\infty\},$$

with $(\eta_k)_k$ and $(\nu_k)_k$ sequences of quasi–eigenvalues of $-\Delta_p$ in $W_0^{1,p}(\Omega)$ then

(P_∞) has at least $k - h + 1$ distinct pairs of nontrivial solutions

Previous results: $l_0 + l_\infty = 0$ (Li-Zhou) or also $l_0 + l_\infty \notin \sigma(-\Delta_p)$ (Perera-Szulkin)

About $\sigma(-\Delta_p)$

The spectral properties of the p-Laplacian in $W_0^{1,p}(\Omega)$ are still mostly unknown

- eigenvalues $(\mu_k)_k$ in García-Peral 1987 via the Krasnoselskii genus
  eigenvalues $(\mu'_k)_k$ in Perera-Szulkin 2005 via the cohomological index of Fadell and Rabinowitz
  $(\mu_k)_k$ and $(\mu'_k)_k$ are unbounded, increasing and $\mu'_k \geq \mu_k$
- the first eigenvalue is characterized as

$$\mu_1 = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p \, \mathrm{d}x}{\int_\Omega |u|^p \, \mathrm{d}x}$$

(positive, simple, isolated and has a positive eigenfunction φ_1)

eigenvalues do not provide for $W_0^{1,p}(\Omega)$ a decomposition similar to that of $W_0^{1,2}(\Omega)$

Quasi–eigenvalues for $-\Delta_p$

In Candela-Palmieri 2009:
$(\eta_h)_h$ increasing and diverging sequence
corresponding functions $(\psi_h)_h$ generate the whole $W_0^{1,p}(\Omega)$
$\psi_1 \equiv \varphi_1$, $\eta_1 = \mu_1$

$$W_0^{1,p}(\Omega) = V_h \oplus W_h \quad \text{for all } h \in \mathbb{N}, h \geq 1$$

where $V_h = \mathrm{span}\{\psi_1, \ldots, \psi_h\}$

$$\eta_{h+1} \int_\Omega |w|^p \, \mathrm{d}x \leq \int_\Omega |\nabla w|^p \, \mathrm{d}x \quad \forall h \in \mathbb{N} \text{ and } w \in W_h$$

if $p = 2$, $(\eta_h)_h$ agrees with $(\lambda_h)_h$

Quasi–eigenvalues for $-\Delta_p$

In Li-Zhou 2002:
$(\nu_k)_k$ increasing and diverging sequence
for all $k \in \mathbb{N}$

$\mathcal{W}_k = \{V : V$ is a subspace of $W_0^{1,p}(\Omega), \varphi_1 \in V, \dim V \geq k, k \in \mathbb{N}\}$

$$\nu_k = \inf_{V \in \mathcal{W}_k} \sup_{u \in V \setminus \{0\}} \frac{\int_\Omega |\nabla u|^p \, \mathrm{d}x}{\int_\Omega |u|^p \, \mathrm{d}x}$$

$\nu_1 = \mu_1$
if $p = 2$, $(\nu_k)_k$ agrees with $(\lambda_k)_k$
for all $k \in \mathbb{N}$: $\nu_k \geq \mu_k$

Main tools of the proof

- Genus (Coffman) and pseudo-index (Benci) related to the genus
- V, W closed subspaces of X; if

$$\dim V < +\infty \quad \text{and} \quad \operatorname{codim} W < +\infty$$

then, for all odd bounded homeomorphism h on X and for all open bounded symmetric neighbourhood $B \subset X$ of 0:

$$\gamma(V \cap h(\partial B \cap W)) \geq \dim V - \operatorname{codim} W$$

- the functional J satisfies a variant of the Palais-Smale condition at level c ($c \in \mathbb{R}$): any sequence $(u_n)_n \subseteq X$ s.t.

$$\lim_{n \to +\infty} I(u_n) = c$$

$$\lim_{n \to +\infty} \|\mathrm{d}I(u_n)\|_{X'}(1 + \|u_n\|_X) = 0$$

converges in X, up to subsequences

The proof

- there exists $V^\sigma \in \mathcal{W}_k$ with $\dim V^\sigma = k$ s.t. (using the ν_k)

$$J(u) \leq c_\infty \qquad \forall u \in V^\sigma$$

- using the assumption on l_0 and η_k setting $S_\rho = \{u \in W_0^{1,p}(\Omega) : \|u\| = \rho\}$ if ρ is small enough

$$J(u) \geq c_0 \qquad \forall u \in S_\rho \cap W_{h-1}$$

- $(S_\rho \cap W_{h-1}, \mathcal{H}^*, \gamma^*)$ pseudo-index theory related to the genus, $S_\rho \cap W_{h-1}$ and J

$$\gamma^*(V^\sigma) = \min_{h \in \mathcal{H}^*} \gamma(V^\sigma \cap h^{-1}(S_\rho \cap W_{h-1})) \geq \dim V^\sigma - \text{ codim } W_{h-1}$$

- Abstract theorem in P.Bartolo-Benci-Fortunato adapted to Banach spaces

Remarks

For all $k \in \mathbb{N}$

$$\eta_k \le \mu_k \le \nu_k$$

Under additional assumptions:

- existence results
- resonant case
- $l_0 \in \{\pm\infty\}$

Results for perturbed problems (also with h only continuous)

$$(P_\varepsilon) \qquad \begin{cases} -\Delta_p u = g(x,u) + \varepsilon h(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$