Quantum Dilations of Linear Cellular Automata

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International Workshop on Functional Analysis
(October 12 - 14, 2012, Timişoara, Romania)

Outline

Cellular Automata

Cellular Automata (CAs)

- introduced by *S. Ulam* and *J. von Neumann*
- homogeneous lattices of cells; their states updated synchronously, according to the state of the neighborhood, by a set of local rules
- *Hedlund*: global function of a CA = continuous map which commutes with the shift operator
- applications in physics, biology, chemistry, cryptography, etc.

Reversible Cellular Automata

Reversible Cellular Automata (RCAs)

- reversibility - one of the most important characteristics of microscopic mechanisms in physics
- CAs can capture reversibility without sacrificing the computational universality, locality of the interactions or simultaneity of the motion
- a CA is reversible if there exists another cellular automaton such that their global transition functions are inverses of each other
- *Hedlund, Richardson*: a CA ($\mathcal{A}$) is reversible iff its global map ($G_{\mathcal{A}}$) is bijective

Quantum Cellular Automata

Quantum Cellular Automata (QCAs)

- *R.P. Feynman* suggested the necessity in quantizing the CA model
- *P. Arrighi, V. Nesme, R.F. Werner*: a 1D QCA - unitary transformation, translation-invariant and causal
- are reversible CAs because their evolution is determined by a unitary transformation
- used in the recent years as quantum mechanical models of computing

Linear Cellular Automata

Definition

- A *linear cellular automaton* (LCA) is a triple $\mathcal{A} = (\mathcal{H}, N, \delta)$ where $\mathcal{H}$ (the *state space*) is a complex Hilbert space (finite or infinite dimensional), $N = N_1 = \{-1, 0, 1\}$ (the *neighborhood*) and $\delta : \mathcal{H}^3 \to \mathcal{H}$ (the *local rule*) is a linear and bounded map.
- The *configuration space* is the Hilbert space $\mathcal{C}_\mathcal{A} := \ell^2_\mathbb{Z}(\mathcal{H})$ of all square summable sequences $(h_n)_{n\in\mathbb{Z}}$ of vectors in $\mathcal{H}$.
- The *global transition function* is defined as

$$\mathcal{C}_\mathcal{A} \ni c = (h_n)_{n\in\mathbb{Z}} \mapsto G_\mathcal{A}(c) := (\delta(h_{n+N}))_{n\in\mathbb{Z}} \in \mathcal{C}_\mathcal{A}.$$

The Structure of the Local Rule

Remark

$$\delta(h_{-1}, h_0, h_1) = \delta_{-1}(h_{-1}) + \delta_0(h_0) + \delta_1(h_1), \qquad h_{-1}, h_0, h_1 \in \mathcal{H},$$

where $\delta_{-1}, \delta_0, \delta_1$ are bounded linear operators on $\mathcal{H}$.
More precisely, they are defined as:

$$\begin{aligned} \delta_{-1}(h) &= \delta(h, 0, 0) \\ \delta_0(h) &= \delta(0, h, 0) \\ \delta_1(h) &= \delta(0, 0, h), \quad h \in \mathcal{H}. \end{aligned}$$

The Adjoint of a Linear Cellular Automaton

Definition

The *adjoint* of a linear cellular automaton $\mathcal{A} = (\mathcal{H}, N, \delta)$ is the LCA $\mathcal{A}^* = (\mathcal{H}, N, \delta_*)$, where

$$\delta_*(h_{-1}, h_0, h_1) := \delta_1^*(h_{-1}) + \delta_0^*(h_0) + \delta_{-1}^*(h_1), \quad h_{-1}, h_0, h_1 \in \mathcal{H},$$

the operators δ_{-1}, δ_0 and δ_1 being defined as in the previous remark.

Definitions and Example

Definition

A LCA is said to be *isometric* (respectively *partial isometric*, respectively *quantum*) if its global transition function is an isometric (respectively partial isometric, respectively unitary) map.

Example

The *right shift* LCA is the LCA $\mathcal{S} = (\mathcal{H}, N, \delta)$, where the local transition function is given by the operators $\delta_{-1} = 1_{\mathcal{H}}$ and $\delta_0 = \delta_1 = 0_{\mathcal{H}}$.

Definitions in Terms of Local Rules

Proposition

A LCA $\mathcal{A} = (\mathcal{H}, N, \delta)$ is isometric if and only if

$$\delta_{-1}^*\delta_1 = \delta_0^*\delta_1 + \delta_{-1}^*\delta_0 = 0_{\mathcal{H}}$$

and

$$\delta_{-1}^*\delta_{-1} + \delta_0^*\delta_0 + \delta_1^*\delta_1 = 1_{\mathcal{H}}.$$

Proposition

A LCA $\mathcal{A} = (\mathcal{H}, N, \delta)$ is a QLCA if and only if

$$\delta_{-1}^*\delta_1 = \delta_1\delta_{-1}^* = \delta_0^*\delta_1 + \delta_{-1}^*\delta_0 = \delta_1\delta_0^* + \delta_0\delta_{-1}^* = 0_{\mathcal{H}}$$

and

$$\delta_{-1}^*\delta_{-1} + \delta_0^*\delta_0 + \delta_1^*\delta_1 = \delta_{-1}\delta_{-1}^* + \delta_0\delta_0^* + \delta_1\delta_1^* = 1_{\mathcal{H}}.$$

The Dilation of a LCA

Definition

Let $\mathcal{A} = (\mathcal{H}, N, \delta)$ and $\mathcal{B} = (\mathcal{K}, N, \varepsilon)$ be two LCAs.

- $\mathcal{B}$ is a *dilation of* $\mathcal{A}$ if $\mathcal{H}$ is a closed subspace of $\mathcal{K}$ and

$$\delta_{i_1}\delta_{i_2}\ldots\delta_{i_n}h = P^{\mathcal{K}}_{\mathcal{H}}\varepsilon_{i_1}\varepsilon_{i_2}\ldots\varepsilon_{i_n}h,$$

  for every $n \in \mathbb{N}^*,\ i_1, i_2, \ldots, i_n \in N$ and $h \in \mathcal{H}$;
- $\mathcal{B}$ is an *extension of* $\mathcal{A}$ if $\mathcal{H}$ is a closed subspace of $\mathcal{K}$ and ε is an extension of δ.

Dilatability to Isometric LCA

Theorem

Let $\mathcal{A} = (\mathcal{H}, N, \delta)$ be a LCA. If

$$\delta_{-1}\delta_{-1}^* + \delta_0\delta_0^* + \delta_1\delta_1^* \leq 1_{\mathcal{H}}$$

then there exists a LCA $\mathcal{B} = (\mathcal{K}, N, \varepsilon)$ with the following properties:

(a) $\mathcal{B}^*$ *is an extension of* $\mathcal{A}^*$ *(hence,* $\mathcal{B}$ *is a dilation of* $\mathcal{A}$*).*

(b) $\varepsilon_i^*\varepsilon_j = \delta_{ij}, \quad i, j \in N.$

(c) $\frac{1}{\sqrt{3}}\mathcal{B} := (\mathcal{K}, N, \frac{1}{\sqrt{3}}\varepsilon)$ *is an isometric LCA.*

Power Dilation

Definition

Let $\mathcal{A} = (\mathcal{H}, N_r, \delta)$ and $\mathcal{B} = (\mathcal{K}, N_s, \varepsilon)$ be two LCAs such that $r \leq s$. $\mathcal{B}$ is said to be a *power dilation* of $\mathcal{A}$ if

(a) $\mathcal{H}$ is a closed subspace of $\mathcal{K}$;

(b) $G_{\mathcal{A}}^m(c) = P_{\mathcal{C}_\mathcal{A}}^{\mathcal{C}_\mathcal{B}} G_{\mathcal{B}}^m(c)$, for every $m \geq 0$ and $c \in \mathcal{C}_\mathcal{A}$.

Power Dilatability in Terms of Local Rules

Proposition

Let $\mathcal{A} = (\mathcal{H}, N_r, \delta)$ and $\mathcal{B} = (\mathcal{K}, N_s, \varepsilon)$ be two LCAs such that $\mathcal{H}$ is a closed subspace of $\mathcal{K}$ and $r \leq s$. Then $\mathcal{B}$ is a power dilation of $\mathcal{A}$ if and only if

$$\sum_{\substack{i_1+\cdots+i_m=l \\ i_1,\ldots,i_m\in N_r}} \delta_{i_1}\ldots\delta_{i_m}(h) = P_{\mathcal{H}}^{\mathcal{K}} \sum_{\substack{i_1+\cdots+i_m=l \\ i_1,\ldots,i_m\in N_s}} \varepsilon_{i_1}\ldots\varepsilon_{i_m}(h), \qquad m \geq 1,\ l \in N_{rm},\ h \in \mathcal{H}$$

and

$$\sum_{\substack{i_1+\cdots+i_m=l \\ i_1,\ldots,i_m\in N_s}} \varepsilon_{i_1}\ldots\varepsilon_{i_m}(h) \perp \mathcal{H}, m \geq 1, \quad l \in N_{sm} \setminus N_{rm},\ h \in \mathcal{H}.$$

Dilatability Implies Power Dilatability

The connection between the notions of dilatability and power dilatability follows as a consequence:

Corollary

Let $\mathcal{A} = (\mathcal{H}, N_r, \delta)$ and $\mathcal{B} = (\mathcal{K}, N_r, \varepsilon)$ be two LCAs. If $\mathcal{B}$ is a dilation of $\mathcal{A}$ then it is also a power dilation of $\mathcal{A}$.

Corollary

Let $\mathcal{A} = (\mathcal{H}, N, \delta)$ be a LCA such that δ is a row contraction. Then $\mathcal{A}$ has a power dilation $\mathcal{B}$ such that $\frac{1}{\sqrt{3}} G_{\mathcal{B}}$ is an isometric operator (if we consider neighborhoods of radius r then $\frac{1}{\sqrt{3}} G_{\mathcal{B}}$ must be replaced by $\frac{1}{\sqrt{2r+1}} G_{\mathcal{B}}$).

Dilatability to Quantum Linear Cellular Automata

Our main theorems are (power) dilatability results for partial isometric LCAs:

Theorem

Let $\mathcal{A} = (\mathcal{H}, N, \delta)$ be a partial isometric LCA. Then there exists an isometric LCA $\mathcal{B} = (\mathcal{K}, N_2, \varepsilon)$ which power dilates $\mathcal{A}$.

Corollary

Any isometric LCA $\mathcal{A}$ can be power dilated to a quantum LCA $\mathcal{B}$ such that $G_{\mathcal{A}}$ is extended by $G_{\mathcal{B}}$.

Theorem

Any partial isometric LCA can be power dilated to a quantum LCA.

Thank you for your attention!